

Dirac operators twisted by ramified Euclidean line bundles

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Abstract

This article is concerned with the analysis of Dirac operators D twisted by ramified Euclidean line bundles (Z, \mathbb{I}) —motivated by their relation with harmonic $\mathbb{Z}/2\mathbb{Z}$ spinors, which have appeared in various context in gauge theory and calibrated geometry. The closed extensions of D are described in terms of the Gelfand–Robbin quotient \check{H} . Assuming that the branching locus Z is a closed cooriented codimension two submanifold, a geometric realisation of \check{H} is constructed. This, in turn, leads to an L^2 regularity theory.

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1 Introduction

Let (X, g) be a closed Riemannian manifold of dimension n .

Definition 1.1 (cf. [LM89, Chapter II Definition 5.2; Bis89, §1(b)]). A **Dirac bundle with skew torsion** on (X, g) consists of:

- (1) a Euclidean vector bundle S over X equipped with a skew-adjoint Clifford multiplication $\gamma: TX \rightarrow \mathfrak{o}(S)$; that is:

$$\gamma(v)^2 = -|v|^2 1_S$$

for every $v \in TX$; and

- (2) an orthogonal covariant derivative ∇ on S and a 3-form $\text{Tor} \in \Omega^3(X)$ such that γ is parallel with respect to ∇ and the orthogonal affine connection ∇^T on (X, g) defined by

$$\langle \nabla_u^T v, w \rangle = \langle \nabla_u^{\text{LC}} v, w \rangle + \frac{1}{2} \text{Tor}(u, v, w).$$

Here ∇^{LC} denotes the Levi-Civita connection of (X, g) . •

Definition 1.2. A **ramified Euclidean line bundle** over X consists of:

- (1) a closed subset $Z \subset X$, the **branching locus**, and
- (2) a Euclidean line bundle \mathbb{I} over $X \setminus Z$

such that

- (3) if $W \subset Z$ is closed and \mathbb{I} extends over $X \setminus W$, then $W = Z$. •

This article is concerned with the analysis of the Dirac operator associated with a Dirac bundle with skew torsion $(S, \gamma, \nabla, \text{Tor})$ twisted by a ramified Euclidean line bundle \mathbb{I}

$$D: H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$$

and its extensions.

The authors' motivation for this stems from the following. Taubes has observed that the failure of compactness for a wide variety of generalised Seiberg–Witten equations—e.g.: stable flat $\text{PSL}_2(\mathbb{C})$ -connections in dimension three [Tau13a], anti-self-dual $\text{SL}_2(\mathbb{C})$ -connections in dimension four [Tau13b], the Seiberg–Witten equation with multiple spinors [Tau16], the Vafa–Witten equation [Tau17], and the Kapustin–Witten equation [Tau22]—leaves behind evidence in the form of a **harmonic $\mathbb{Z}/2\mathbb{Z}$ spinor**. The latter is a pair $(Z, \mathbb{I}; \Phi)$ consisting of a ramified Euclidean line bundle (Z, \mathbb{I}) and a harmonic spinor $\Phi \in \ker D$. Harmonic $\mathbb{Z}/2\mathbb{Z}$ spinors also appear in Donaldson's work on adiabatic limits of coassociative Kovalev–Lefschetz fibrations

of G_2 -manifolds [Don17] and He's work on branched double covers of special Lagrangian submanifolds [He22].

In light of this, it is important to understand the universal moduli space of harmonic $\mathbb{Z}/2\mathbb{Z}$ spinors (allowing for g , γ , and ∇ to vary). The fundamental issue is that D is only left semi-Fredholm (under mild assumptions; see Hypothesis 2.1), but not Fredholm—except in edge cases, e.g., if $Z = \emptyset$ or $n = 2$. The naive expectation is that the ∞ -dimensional cokernel of D can be compensated by wiggling the branching locus Z . In his PhD thesis, Takahashi [Tak15; Tak17] has made some initial progress in this direction. Donaldson [Don21] and Parker [Par23] have developed a (partial) deformation theory for harmonic $\mathbb{Z}/2\mathbb{Z}$ 1-forms and harmonic $\mathbb{Z}/2\mathbb{Z}$ spinors on spin 3-manifolds respectively. There is work in progress by He, Parker and Walpuski to address this problem a bit more systematically. The present article should be considered infrastructure for this project (and, hopefully, other applications as well).

Here is a summary of the results contained in this article. Section 2 considers D as an unbounded operator D_{\min} on $L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$, the **minimal extension**, and systematically studies its closed extensions. The adjoint $D_{\max} := D_{\min}^*$ is the **maximal extension** of D_{\min} . The closed extensions D_B of D_{\min} are classified by **residue conditions**, that is: closed subspaces $B \subset \check{H}$ of the **Gelfand–Robbin quotient**

$$\check{H} := \frac{\text{dom}(D_{\max})}{\text{dom}(D_{\min})}.$$

Moreover, \check{H} is equipped with a symplectic form G , the **Green's form**, which controls the formation of adjoints. Within this framework it is also possible to describe which extensions D_B are Fredholm. The entire discussion only relies on D_{\min} being closed, densely defined, and symmetric as well as left semi-Fredholm. It is confined to the realm of abstract functional analysis and its purpose is to separate what is true for formal reasons from what is true for geometric reasons. Most of the observations in Section 2 can be found in [MS98, Exercise 2.17; BF98, §3; SWo8, Appendix B; BS18, Exercises 6.3.3 and 6.5.11] in some shape or form.

Assuming that $Z \subset X$ is a closed (cooriented) submanifold of codimension two, Section 3 constructs an isomorphism of symplectic Hilbert spaces

$$\text{res}: (\check{H}, G) \cong (\check{H}\Gamma(Z, \check{S}), \check{\Omega}),$$

the **residue map**, between the Gelfand–Robbin quotient and a Hilbert space of sections of a symplectic vector bundle over Z . The residue map extracts the leading order behavior of $\phi \in \text{dom}(D_{\max})$ which is shown to be (at worst) comparable to $\bar{z}^{-1/2}$ transversely to Z . With the help of the above it is possible to define spectral residue conditions, analogous to the APS boundary condition [APS75], as well as local residue conditions. As by product this yields a variant of the bordism theorem, whose significance remains somewhat mysterious to the authors. Evidently the construction in Section 3 is inspired by Bär and Ballmann's magnificent article [BB12] on boundary value problems for Dirac operators.

Finally, Section 4 develops an L^2 regularity theory on the scale of **adapted Sobolev spaces** $(H_a^k\Gamma(X \setminus Z, S \otimes \mathbb{I}))_{k \in \mathbb{N}_0}$. This scale is defined via the ring of differential operators generated by conormal differential operators and the Dirac operator D . It gives rise to a graded Fréchet space

$H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I})$ which is tame in the sense of Hamilton [Ham82, Part II Definition 1.3.2]—a prerequisite for using Nash–Moser theory. Moreover, spinors $\phi \in H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I})$ extend smoothly across Z after untwisting by $\bar{z}^{1/2}$ and, therefore, have well-behaved polyhomogeneous expansions near Z . Crucially, it is proved that if a residue condition $B \subset \check{H}\Gamma(Z, \check{S})$ is ∞ –**regular**, then the extension D_B satisfies a variant of elliptic regularity together with elliptic estimates. For local residue conditions B , ∞ –regularity can be verified using straight-forward symbolic criterion. In particular, this criterion applies to the Lagrangian local residue condition which is secretly at the heart of [Tak15; Par23]. It is quite plausible that these results can be cobbled together using the powerful machines developed by Mazzeo [Maz91], Mazzeo and Vertman [MV14], and Albin and Gell-Redman [AG16; AG23]. However, the arguments in Section 4 are almost elementary and there should be some value in that.

It should be possible, with suitable modifications, to extend the work in the present article to higher rank ramified Euclidean local systems; in particular: to flat Hermitian line bundles. In fact, Ammann–Große have on-going work in progress in this direction and some instances of this appear in Portmann, Sok, and Solovej’s work on magnetic links [PSS18b; PSS18a; PSS20].

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Conventions. Choose a cut-off function $\chi \in C^\infty([0, \infty), [0, 1])$ with $\chi|_{[0, 1/4]} = 1$ and $\text{supp}(\chi) \subset [0, 1/2)$. The bracket $\langle - \rangle : \mathbb{R} \rightarrow [1, \infty)$ is defined by $\langle x \rangle := (1 + x^2)^{1/2}$.

2 The Gelfand–Robbin quotient, I: abstract theory

This section studies the closed extensions of the Dirac operator $D : H^1 \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I})$, considered as an unbounded operator, following [MS98, Exercise 2.17; BF98, §3; SW08, Appendix B; BS18, Exercises 6.3.3 and 6.5.11]. Throughout, assume the following analytic condition on the branching locus Z .

Hypothesis 2.1. *There is an $r \in C^\infty(X \setminus Z, (0, \infty))$, uniformly comparable to the Riemannian distance to Z , such that following **borderline Hardy inequality** holds: for every $\phi \in H^1 \Gamma(X \setminus Z, S \otimes \mathbb{I})$, $r^{-1}\phi \in L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I})$ and*

$$\|r^{-1}\phi\|_{L^2} \lesssim \|\phi\|_{H^1}.$$

Remark 2.2. Hypothesis 2.1 holds if Z is a codimension two submanifold; see Takahashi [Tak15, Lemma 2.6] or Lemma 3.3. Moreover, it holds in the situation considered by Haydys, Mazzeo, and Takahashi [HMT23] where $Z \subset X$ is a graph embedded in a 3–manifold. ♣

2.1 The minimal and maximal extension

Proposition 2.3. *The bounded operator $D: H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is left semi-Fredholm; that is: $\ker D$ is finite-dimensional and $\operatorname{im} D$ is closed.*

The proof relies on the following consequences of the borderline Hardy inequality.

Lemma 2.4. *The following hold:*

- (1) $H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) = H_0^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$.
- (2) *The inclusion $H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \hookrightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is a compact operator.*

Proof. For $\varepsilon > 0$ set $\chi_\varepsilon := \chi(r/\varepsilon)$. Let $\phi \in H^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$. Since $|\operatorname{rd}\chi_\varepsilon| \lesssim 1$,

$$\|\nabla(\chi_\varepsilon\phi)\|_{L^2} \leq \|(\operatorname{rd}\chi_\varepsilon)r^{-1}\phi\|_{L^2} + \|\chi_\varepsilon\nabla\phi\|_{L^2} \lesssim \left(\int_{\operatorname{supp}(\operatorname{d}\chi_\varepsilon)} |r^{-1}\phi|^2 + |\nabla\phi|^2 \right)^{1/2}.$$

Therefore, by [Hypothesis 2.1](#) and monotone convergence,

$$\lim_{\varepsilon \downarrow 0} \|\nabla(\chi_\varepsilon\phi)\|_{L^2} = 0.$$

This implies (1).

Let $(\phi_n) \in H^1\Gamma(X \setminus Z, S \otimes \mathbb{I})^{\mathbb{N}}$ with $\|\phi_n\|_{H^1} = 1$ for every $n \in \mathbb{N}$. For every $\varepsilon > 0$, a subsequence of $((1 - \chi_\varepsilon)\phi_n)$ converges in $L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$. By the borderline Hardy inequality,

$$\|\chi_\varepsilon\phi_n\|_{L^2} \lesssim \varepsilon.$$

Therefore, the (2) follows from a diagonal sequence argument. ■

The proof of [Proposition 2.3](#) also uses the following observation.

Proposition 2.5 (cf. Bismut [[Bis89](#), Theorem 1.10]).

- (1) *D is formally self-adjoint; in fact: for every $\phi, \psi \in H_{\operatorname{loc}}^1\Gamma(X, S \otimes \mathbb{I})$*

$$\langle D\phi, \psi \rangle - \langle \phi, D\psi \rangle = \operatorname{div}(v) \quad \text{with} \quad v := \sum_{i=1}^n \langle \gamma(e_i)\phi, \psi \rangle \cdot e_i.$$

Here (e_1, \dots, e_n) denotes a local orthonormal frame.

- (2) *D satisfies*

$$D^2 = \nabla^*\nabla + \tau\nabla + \gamma(F_\nabla)$$

with $\tau \in \Gamma(X, \operatorname{Hom}(T^*X \otimes S, S))$ depending linearly on Tor .

Proof. The following argument can be found in [[Bis89](#), Proof of Theorem 1.10] and is repeated here only for the readers' convenience.

By direct computation,

$$\operatorname{div} v = \sum_{i=1}^n \mathcal{L}_{e_i} \langle v, e_i \rangle = \langle D\phi, \psi \rangle - \langle \phi, D\psi \rangle + \sum_{i=1}^n \langle \gamma(\nabla_{e_i}^T e_i)\phi, \psi \rangle$$

and

$$\langle \nabla_{e_i}^T e_i, - \rangle = \frac{1}{2} \text{Tor}(e_i, e_i, -) = 0.$$

This proves (1).

By direct computation,

$$\begin{aligned} D^2 &= \sum_{i,j=1}^n \gamma(e_i) \nabla_{e_i} \gamma(e_j) \nabla_{e_j} = \sum_{i,j=1}^n \gamma(e_j) \gamma(e_i) \nabla_{e_i} \nabla_{e_j} + \gamma(e_i) \gamma(\nabla_{e_i}^T e_j) \nabla_{e_j} \\ &= \nabla^* \nabla + \gamma(F_\nabla) + \gamma(e_i) \gamma(\nabla_{e_i}^T e_j) \nabla_{e_j} \end{aligned}$$

and

$$\langle \nabla_{e_i}^T e_j, e_k \rangle = \frac{1}{2} \text{Tor}(e_i, e_j, e_k).$$

This proves (2). ■

Proof of Proposition 2.3. By Lemma 2.4 (1) and Proposition 2.5,

$$(2.6) \quad \|\phi\|_{H^1} \lesssim \|D\phi\|_{L^2} + \|\phi\|_{L^2}$$

for every $\phi \in H^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$. Therefore and by Lemma 2.4 (2), D is left semi-Fredholm. ■

With the exception of a few edge cases—e.g.: if $Z = \emptyset$ or $Z \subset X$ is a finite subset of a surface [DW24, §3.4.2; HMT23, §4.1–4.5]—the operator $D: H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is not Fredholm: its cokernel is ∞ -dimensional. Therefore, it is useful to consider D as an unbounded operator and systematically study its closed extensions; cf. [BS18, Chapter 6].

Definition 2.7. The **minimal extension**

$$D_{\min}: \text{dom}(D_{\min}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I});$$

is the operator $D: H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ considered as unbounded operator on $L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$. •

Proposition 2.8. D_{\min} is closed, densely defined, and symmetric.

Proof. Evidently, D_{\min} is densely defined. By Proposition 2.5 (1),

$$\langle D\phi, \psi \rangle = \langle \phi, D\psi \rangle$$

for every $\phi, \psi \in \Gamma_c(X \setminus Z, S \otimes \mathbb{I})$. Therefore, by Lemma 2.4 (1) and since $\Gamma_c(X \setminus Z, S \otimes \mathbb{I}) \subset H_0^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is dense, D_{\min} is symmetric. By (2.6) the Sobolev norm $\|-\|_{H^1}$ and the graph norm $\|-\|_D := (\|-\|_{L^2}^2 + \|D-\|_{L^2}^2)^{1/2}$ are equivalent. Therefore and since $H^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is complete, D_{\min} is closed. ■

Definition 2.9. The **maximal extension**

$$D_{\max}: \text{dom}(D_{\max}) \rightarrow L^2(X \setminus Z, S \otimes \mathbb{I})$$

is the adjoint of D_{\min} in the sense of unbounded operators; that is:

$$\text{dom}(D_{\max}) := \{\phi \in L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) : \langle \phi, D_{\min} - \rangle_{L^2} : \text{dom}(D_{\min}) \rightarrow \mathbb{R} \text{ is } \|\cdot\|_{L^2}\text{-bounded}\}$$

and for every $\phi \in \text{dom}(D_{\max})$ and $\psi \in \text{dom}(D_{\min})$

$$\langle D_{\max}\phi, \psi \rangle = \langle \phi, D_{\min}\psi \rangle.$$

$D_{\max}\phi$ exists by the Hahn–Banach Theorem and the Riesz Representation Theorem, and is unique because $\text{dom}(D_{\min})$ is dense. •

Remark 2.10. It is convenient to consider $D : H_{\text{loc}}^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow L_{\text{loc}}^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$. From this perspective,

$$\text{dom}(D_{\max}) = \{\phi \in H_{\text{loc}}^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) : \phi, D\phi \in L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})\};$$

and, it is excusable to drop the subscripts from $D_{\min}\phi$, $D_{\max}\phi$, etc. ♣

2.2 Closed extensions and residue conditions

The closed extensions of D_{\min} can be systematically understood as follows.

Definition 2.11. The **Gelfand–Robbin quotient** is the Hilbert space

$$\check{H} := \frac{\text{dom}(D_{\max})}{\text{dom}(D_{\min})}.$$

Since D_{\min} is closed, $\text{dom}(D_{\min}) \subset \text{dom}(D_{\max})$ is a $\|\cdot\|_D$ -closed subspace. Denote the canonical projection map by

$$[\cdot] : \text{dom}(D_{\max}) \rightarrow \check{H}. \quad \bullet$$

Remark 2.12. \check{H} is localised on Z in the following sense: $[\phi] = [\chi(r/\varepsilon)\phi]$ for every $\varepsilon > 0$ and $\phi \in \text{dom}(D_{\max})$. ♣

Definition 2.13. A **residue condition** is a closed subspace $B \subset \check{H}$. •

Proposition 2.14 (closed extension=residue condition; cf. [BF98, Lemma 3.3(a)]). *If $B \subset \check{H}$ is a residue condition, then*

$$D_B := D_{\max}|_{\text{dom}(D_B)} \quad \text{with} \quad \text{dom}(D_B) := [\cdot]^{-1}(B)$$

is a closed extension of D_{\min} . Moreover, every closed extension of D_{\min} is of this form.

Proof. Let $B \subset \check{H}$ be a residue condition. The canonical projection $[\cdot] : \text{dom}(D_{\max}) \rightarrow \check{H}$ is bounded. Therefore, $\text{dom}(D_B) := [\cdot]^{-1}(B) \subset \text{dom}(D_{\max})$ is a $\|\cdot\|_D$ -closed subspace; hence: D_B is closed.

Let \bar{D} be a closed extension of D_{\min} . Since $\text{dom}(\bar{D}) \subset \text{dom}(D_{\max})$ is a $\|\cdot\|_D$ -closed subspace, $B := [\text{dom}(\bar{D})] = \frac{\text{dom}(\bar{D})}{\text{dom}(D_{\min})} \subset \check{H}$ is a closed subspace. Since $\text{dom}(D_{\min}) \subset \text{dom}(\bar{D})$, $\text{dom}(\bar{D}) = [\cdot]^{-1}(B)$; hence: $\bar{D} = D_B$. ■

2.3 The Green's form and adjoint extensions

The Gelfand–Robbin quotient carries a symplectic structure related to the construction of adjoint extensions.

Definition 2.15. The **Green's form** $G \in \text{Hom}(\Lambda^2 \check{\mathbf{H}}, \mathbf{R})$ is defined by

$$G([\phi] \wedge [\psi]) := \langle D\phi, \psi \rangle_{L^2} - \langle \phi, D\psi \rangle_{L^2}. \quad \bullet$$

Proposition 2.16 (cf. [BF98, Lemma 3.1, Proposition 3.2; SW08, Remark B.1(ii)]). G is symplectic; that is: it induces a Hilbert space isomorphism

$$\begin{aligned} J: \check{\mathbf{H}} &\rightarrow \hat{\mathbf{H}} := \mathcal{L}(\check{\mathbf{H}}, \mathbf{R}) \\ [\phi] &\mapsto G([\phi] \wedge -). \end{aligned}$$

Moreover: if $\sharp: \hat{\mathbf{H}} \rightarrow \check{\mathbf{H}}$ denotes the isomorphism induced by the inner product, then $\sharp \circ J$ is an isometric complex structure.

Proof. The canonical projection induces an isometry $[-]: \text{dom}(D_{\min})^{\perp D} \cong \check{\mathbf{H}}$. Here \perp_D indicates the orthogonal complement with respect to the graph inner product $\langle -, - \rangle_D := \langle -, - \rangle_{L^2} + \langle D-, D- \rangle_{L^2}$. By direct inspection,

$$\begin{aligned} \text{dom}(D_{\min})^{\perp D} &= \{\phi \in \text{dom}(D_{\max}) : \langle \phi, \psi \rangle_{L^2} + \langle D\phi, D\psi \rangle_{L^2} = 0 \text{ for every } \psi \in \text{dom}(D_{\min})\} \\ &= \{\phi \in \text{dom}(D_{\max}) : D\phi \in \text{dom}(D_{\max}) \text{ and } D^2\phi = -\phi\}. \end{aligned}$$

Therefore, D induces an isometric complex structure $D: \text{dom}(D_{\min})^{\perp D} \rightarrow \text{dom}(D_{\min})^{\perp D}$.

The diagram

$$\begin{array}{ccc} \text{dom}(D_{\min})^{\perp D} & \xrightarrow{D} & \text{dom}(D_{\min})^{\perp D} \\ [-] \downarrow & & \downarrow [-] \\ \check{\mathbf{H}} & \xrightarrow{\sharp \circ J} & \check{\mathbf{H}} \end{array}$$

commutes because for every $\phi, \psi \in \text{dom}(D_{\min})^{\perp D}$

$$G([\phi] \wedge [\psi]) = \langle D\phi, \psi \rangle_{L^2} - \langle \phi, D\psi \rangle_{L^2} = \langle D\phi, \psi \rangle_{L^2} + \langle D^2\phi, D\psi \rangle_{L^2} = \langle D\phi, \psi \rangle_D.$$

This proves the assertion. ■

Proposition 2.17 (cf. [BF98, Lemma 3.3(b)]). Let $B \subset \check{\mathbf{H}}$ be a residue condition. The adjoint D_B^* of D_B is the closed extension D_{B^G} associated with the symplectic complement

$$B^G := \{[\phi] \in \check{\mathbf{H}} : G([\phi], [\psi]) = 0 \text{ for every } [\psi] \in B\}.$$

In particular, D_B is self-adjoint if and only if B is Lagrangian.

Proof. A moment's thought shows that

$$\begin{aligned} \text{dom}(D_B^*) &= \{\phi \in \text{dom}(D_{\max}) : \langle D\phi, \psi \rangle_{L^2} = \langle \phi, D\psi \rangle_{L^2} \text{ for every } \psi \in \text{dom}(D_B)\} \\ &= \{\phi \in \text{dom}(D_{\max}) : G([\phi] \wedge [\psi]) = 0 \text{ for every } [\psi] \in B\}. \end{aligned}$$

This proves the assertion. ■

Example 2.18. The Calderón subspace

$$\Lambda := [\ker D_{\max}] \subset \check{H}$$

is a Lagrangian residue condition. Indeed, $\Lambda \subset \Lambda^G$ because $G([\phi] \wedge [\psi]) = \langle D\phi, \psi \rangle_{L^2} - \langle \phi, D\psi \rangle_{L^2} = 0$ for every $\phi, \psi \in \ker D_{\max}$. Moreover, if $[\phi] \in \Lambda^G$, then $D\phi \perp_{L^2} \ker D_{\max} = (\operatorname{im} D_{\min})^{\perp_{L^2}}$; therefore, there is a $\psi \in \operatorname{dom}(D_{\min})$ with $D\psi = D\phi$; hence: $[\phi] = [\phi - \psi] \in \Lambda$. ♠

Example 2.19. As a consequence of [Proposition 2.16](#), the orthogonal complement of the Calderón subspace

$$\Lambda^\perp = \sharp \circ J(\Lambda) \subset \check{H}$$

is a Lagrangian residue condition. ♠

Example 2.20. Suppose that S carries a parallel orthogonal complex structure i which commutes with γ ; that is: (S, γ, ∇) is a **complex Dirac bundle with skew torsion**. Evidently, D is complex linear and \check{H} inherits i as an isometric complex structure i . This induces an orthogonal decomposition

$$\check{H} = B_+ \oplus B_- \quad \text{with} \quad B_\pm := \{[\phi] \in \check{H} : \sharp \circ J[\phi] = \pm i[\phi]\}.$$

Since i and $\sharp \circ J$ commute, $B_\pm \subset \check{H}$ are complex subspaces and, therefore, B_\pm are mutually adjoint:

$$B_\pm^G = B_\mp. \quad \spadesuit$$

Remark 2.21 (defect indices). If H is a *complex* Hilbert space and $A: \operatorname{dom}(A) \rightarrow H$ is a closed and symmetric unbounded *complex* linear operator, then its closed *complex linear* extensions traditionally are studied via von Neumann's theory of **defect subspaces** and **defect indices** [[vNeu30](#), Kapitel VII; [RS75](#), §X.1]. The defect subspaces of A are $\ker(A^* \mp i)$ and its defect indices are $n_\pm := \dim \ker(A^* \mp i)$. The maximal domain orthogonally decomposes as

$$\operatorname{dom}(A^*) = \operatorname{dom}(A) \oplus \ker(A^* - i) \oplus \ker(A^* + i)$$

with respect to the graph inner product. Therefore, $\check{H} := \frac{\operatorname{dom}(A^*)}{\operatorname{dom}(A)} \cong \ker(A^* - i) \oplus \ker(A^* + i)$. In particular, closed self-adjoint complex linear extension of A correspond to closed complex linear Lagrangian subspaces $B \subset \ker(A^* - i) \oplus \ker(A^* + i)$. The latter exist if and only if $n_+ = n_-$. Of course, by Zorn's Lemma, \check{H} always has a (real) Lagrangian subspace. ♣

Proposition 2.22 (Spectral theory). *Let $B \subset \check{H}$ be a residue condition. If $B \subset \check{H}$ is a Lagrangian and $\operatorname{dom}(D_B) \hookrightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is compact, then $\operatorname{spec}(D_B)$ consists only of point spectrum, is contained in \mathbb{R} and discrete, and for every $\lambda \in \operatorname{spec}(D_B)$ the eigenspace $\ker(D_B - \lambda \cdot 1)$ is finite-dimensional; moreover: $L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ decomposes as a (Hilbert space) direct sum*

$$L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) = \bigoplus_{\lambda \in \operatorname{spec}(D_B)} \ker(D_B - \lambda \cdot 1).$$

Proof. By assumption, D_B is self-adjoint and has compact resolvent. The assertion, therefore, follows from the spectral theory of such operators; see, e.g., [[BS18](#), Theorem 6.3.13]. ■

2.4 Fredholm extensions

The following characterises residue conditions $B \subset \check{H}$ which correspond to Fredholm extensions D_B in terms of the relation between B and the Calderón subspace Λ .

Definition 2.23. Let $B \subset \check{H}$ be a residue condition. Denote by

$$\delta_B: \Lambda \rightarrow \check{H}/B \quad \text{and} \quad \delta^B: B \rightarrow \check{H}/\Lambda$$

the compositions of the canonical inclusions and projections. •

Proposition 2.24 (cf. [SW08, Lemma B.3]). *Let $B \subset \check{H}$ be a residue condition. The closed extension D_B is Fredholm if and only if δ_B is Fredholm if and only if δ^B is Fredholm; moreover:*

$$\text{index } D_B = \text{index } \delta_B = \text{index } \delta^B.$$

The proof relies on the following observation.

Lemma 2.25. *For every residue condition $B \subset \check{H}$, there are short exact sequences*

$$\ker D_{\min} \hookrightarrow \ker D_B \twoheadrightarrow \ker \delta_B \quad \text{and} \quad \text{coker } \delta_B \hookrightarrow \text{coker } D_B \twoheadrightarrow \text{coker } D_{\max}.$$

Proof. The Snake Lemma applied to

$$\begin{array}{ccc} \ker D_{\min} & \xlongequal{\quad} & \ker D_{\min} \\ \downarrow & & \downarrow \\ \ker D_B & \hookrightarrow & \ker D_{\max} \xrightarrow{\delta_B \circ [-]} \check{H}/B \end{array}$$

yields an exact sequence

$$\frac{\ker D_B}{\ker D_{\min}} \hookrightarrow \frac{\ker D_{\max}}{\ker D_{\min}} \cong \Lambda \xrightarrow{\delta_B} \check{H}/B.$$

This induces the first short exact sequence.

The Snake Lemma applied to

$$\begin{array}{ccccc} \text{dom}(D_B) & \hookrightarrow & \text{dom}(D_{\max}) & \twoheadrightarrow & \check{H}/B \\ \downarrow D_B & & \downarrow D_{\max} & & \\ L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) & \xlongequal{\quad} & L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) & & \end{array}$$

yields an exact sequence

$$\ker D_B \hookrightarrow \ker D_{\max} \xrightarrow{\delta_B \circ [-]} \check{H}/B \twoheadrightarrow \text{coker } D_B \twoheadrightarrow \text{coker } D_{\max}.$$

Since $\text{coker } \delta_B \circ [-] = \text{coker } \delta_B$, this induces the second short exact sequence. ■

Proof of Proposition 2.24. A moment's thought shows that

$$\ker \delta_B = \Lambda \cap B = \ker \delta^B \quad \text{and} \quad \text{coker } \delta_B \cong \frac{\check{\mathbf{H}}}{\Lambda + B} \cong \text{coker } \delta^B.$$

By Proposition 2.3, $\ker D_{\min} \cong \text{coker } D_{\max}$ is finite-dimensional. Therefore, the assertion is an immediate consequence of Lemma 2.25. \blacksquare

Example 2.26. Every complement B of the Calderón subspace Λ , in particular: $B = \Lambda^\perp$, produces a Fredholm extension of index 0 because $\delta^B : B \rightarrow \check{\mathbf{H}}/\Lambda \cong B$ is an isomorphism. \spadesuit

Example 2.27. The residue conditions B_\pm defined in Example 2.20 satisfy

$$\ker D_{B_\pm} = \ker D_{\min}$$

and, therefore, produce Fredholm extension of index 0; indeed: if $\phi \in \ker D_{B_\pm}$, then

$$0 = 2\langle D\phi, i\phi \rangle_{L^2} = \langle D\phi, i\phi \rangle_{L^2} - \langle \phi, Di\phi \rangle_{L^2} = G([\phi], i[\phi]) = \langle \sharp \circ J[\phi], i[\phi] \rangle_{L^2} = \pm \|\phi\|_{L^2}^2. \quad \spadesuit$$

The following are occasionally useful to compute or relate indices.

Proposition 2.28 (Nested Fredholm residue conditions). *Let $B_1 \subset B_2 \subset \check{\mathbf{H}}$ be residue conditions. If $\delta_{B_1}, \delta_{B_2}$ are Fredholm, then*

$$\text{index } D_{B_2} = \text{index } D_{B_1} + \dim B_2/B_1.$$

Corollary 2.29. *Let $B \subset \check{\mathbf{H}}$ be a residue condition. If $B \subset B^G$ and δ_B is Fredholm then $\text{index } D_B = -\frac{1}{2} \dim B^G/B$; in particular: if B is Lagrangian, then $\text{index } D_B = 0$.* \blacksquare

Proposition 2.28 is an immediate consequence of the following.

Lemma 2.30 (Nested residue conditions). *Let $B_1 \subset B_2 \subset \check{\mathbf{H}}$ be residue conditions. There is an exact sequence*

$$\ker \delta_{B_1} \hookrightarrow \ker \delta_{B_2} \rightarrow B_2/B_1 \rightarrow \text{coker } \delta_{B_1} \twoheadrightarrow \text{coker } \delta_{B_2}.$$

Proof. The exact sequence follows from the Snake Lemma applied to

$$\begin{array}{ccccc} \Lambda & \xlongequal{\quad} & \Lambda & & \\ \downarrow \delta_{B_1} & & \downarrow \delta_{B_2} & & \\ B_2/B_1 & \hookrightarrow & \check{\mathbf{H}}/B_1 & \twoheadrightarrow & \check{\mathbf{H}}/B_2. \end{array}$$

This implies the assertion by the above proposition. \blacksquare

Proposition 2.31 (Deformation of residue conditions). *Let B be a Hilbert space. Let $\iota_- : [0, 1] \rightarrow \mathcal{L}(B, \check{\mathbf{H}})$ be a continuous path of embeddings. If D_{B_t} with $B_t := \iota_t(B)$ is Fredholm for every $t \in [0, 1]$, then*

$$\text{index } D_{B_0} = \text{index } D_{B_1}.$$

Proof. By assumption, $t \mapsto \delta^{B_t} \circ \iota_t$ is a continuous path of Fredholm operators and $\iota_t : B \rightarrow B_t$ is an isomorphism for every $t \in [0, 1]$; therefore: $t \mapsto \text{index } D_{B_t} = \text{index } \delta^{B_t} = \text{index } \delta^{B_t} \circ \iota_t$ is constant. \blacksquare

2.5 Chirality operators

In the presence of a chirality operator the theory discussed above refines as follows.

Definition 2.32. A chirality operator on (S, γ, ∇) is a self-adjoint parallel isometry $\varepsilon \in \Gamma(X, \mathcal{O}(S))$ such that

$$\gamma\varepsilon + \varepsilon\gamma = 0. \quad \bullet$$

Example 2.33. Assume that X is oriented. If $\dim X = 0 \bmod 4$, then $\varepsilon := \gamma(\text{vol}_g)$ is a chirality operator. If $\dim X = 0 \bmod 2$ and (S, γ, ∇) is a complex Dirac bundle as in [Example 2.20](#), then $\varepsilon := i^{\lfloor (n+1)/2 \rfloor} \gamma(\text{vol}_g)$ is a chirality operator. \spadesuit

Proposition 2.34 (Chirality operators induce a $\{\pm 1\}$ -grading). *If ε is a chirality operator for (S, γ, ∇) , then the following hold:*

- (1) *The subbundles $S^\pm := \ker(1 \pm \varepsilon) \subset S$ are parallel, S orthogonally decomposes as*

$$S = S^+ \oplus S^-,$$

and $\gamma \in \Gamma(X, \text{Hom}(TX, \text{Hom}(S^+, S^-) \oplus \text{Hom}(S^-, S^+)))$.

- (2) *The minimal and maximal extensions decompose as*

$$D_{\min} = \begin{pmatrix} 0 & D_{\min}^- \\ D_{\min}^+ & 0 \end{pmatrix} \quad \text{and} \quad D_{\max} = \begin{pmatrix} 0 & D_{\max}^- \\ D_{\max}^+ & 0 \end{pmatrix}$$

with

$$\begin{aligned} D_{\min}^\pm : \text{dom}(D_{\min}^\pm) &:= H^1\Gamma(X \setminus Z, S^\pm \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S^\mp \otimes \mathbb{I}) \quad \text{and} \\ D_{\max}^\pm : \text{dom}(D_{\max}^\pm) &:= \text{dom}(D_{\max}) \cap L^2\Gamma(X \setminus Z, S^\pm \otimes \mathbb{I}) \rightarrow L^2\Gamma(X \setminus Z, S^\mp \otimes \mathbb{I}). \end{aligned}$$

- (3) \check{H} *orthogonally decomposes as*

$$\check{H} = \check{H}^+ \oplus \check{H}^- \quad \text{with} \quad \check{H}^\pm := \frac{\text{dom}(D_{\max}^\pm)}{\text{dom}(D_{\min}^\pm)};$$

moreover, $\check{H}^\pm \subset \check{H}$ are Lagrangian. In particular, every residue condition $B \subset \check{H}$ decomposes as $B = B^+ \oplus B^-$.

- (4) *If $B^+ \subset \check{H}^+$ is a closed subspace, a **positive chirality residue condition**, then there is a unique closed subspace $B^- \subset \check{H}^-$ such that $B := B^+ \oplus B^- \subset \check{H}$ is a Lagrangian residue condition.*

- (5) *Let $B^+ \subset \check{H}^+$ be a **positive chirality residue condition** and B^- as above. The operator δ^B is Fredholm if and only if $\delta^{B^+} : B^+ \rightarrow \check{H}^+/\Lambda^+$ is Fredholm.*

Proof. (1), (2), and (3) are an immediate consequence of [Definition 2.32](#).

A moment's thought shows that (4) holds with $B^- := (B^+)^G \cap \check{H}^-$.

Evidently, $\delta^B = \delta^{B^+} \oplus \delta^{B^-}$ is Fredholm if and only if δ^{B^+} and δ^{B^-} are Fredholm. The Green's form G induces isomorphisms

$$\Lambda^- \cong (\check{H}^+/\Lambda^+)^* \quad \text{and} \quad B^+ \cong (\check{H}^-/B^-)^*.$$

This identifies the dual of δ^{B^+} with $\delta_{B^-} : \Lambda^- \rightarrow \check{H}^-/B^-$. By the closed image theorem, if δ^{B^+} is Fredholm, then δ_{B^-} is Fredholm. As in the proof of [Proposition 2.24](#), δ_{B^-} is Fredholm if and only if δ^{B^-} is Fredholm. This proves (5). \blacksquare

3 The Gelfand–Robbin quotient, II: geometric realisation

The usefulness of the theory laid out in [Section 2](#) hinges upon being able to understand \check{H} , e.g., to specify interesting residue conditions. Since \check{H} localises on Z as explained in [Remark 2.12](#), it is plausible that it admits a more geometric description. The purpose of this section is to develop such a description, assuming the following geometric condition on the branching locus Z throughout.

Hypothesis 3.1. $Z \subset X$ is a closed cooriented submanifold of codimension two.

Remark 3.2. The assumption that Z is cooriented simplifies the upcoming discussion, but is not essential. [Appendix A](#) explains how to remove it. \clubsuit

Lemma 3.3 (borderline Hardy inequality; Takahashi [[Tak15](#), Lemma 2.6]). *Hypothesis 3.1 implies Hypothesis 2.1.*

Proof. Let $r > 0$. Denote by I the non-trivial Euclidean line bundle over $rS^1 := \{z \in \mathbb{C} : |z| = r\}$. A moment's thought and a scaling consideration show that

$$\int_{rS^1} |r^{-1}s|^2 \lesssim \int_{rS^1} |\nabla s|^2$$

for every $s \in \Gamma(rS^1, I)$. This immediately implies the assertion. \blacksquare

3.1 The blow-up of X along Z

It is convenient to blow-up X along Z ; that is: to replace $Z \subset X$ with the following $U(1)$ -principal bundle.

Definition 3.4. Since Z is cooriented, $NZ \rightarrow Z$ is a Hermitian line bundle. Its **frame bundle** is

$$\pi : F := \{v \in NZ : |v| = 1\} \rightarrow Z$$

together with $F \cup U(1)$ defined by $v \cdot e^{i\alpha} := e^{i\alpha}v$. Denote the Levi-Civita connection on F by $i\theta \in \Omega^1(F, i\mathbb{R})$. \bullet

Remark 3.5. The **tautological section** $\partial_r \in \Gamma(F, \pi^*NZ)$ and $\partial_\alpha := i\partial_r$ trivialise π^*NZ . \clubsuit

In order to replace $Z \subset X$ with F a choice is required.

Definition 3.6. Set $U := [0, 1) \cdot F \subset NZ$. A **tubular neighbourhood** $j: U \hookrightarrow X$ of $Z \subset X$ is an embedding such that $j \circ 0 = \text{id}_Z$ and the composition

$$NZ \hookrightarrow 0^*TNZ \xrightarrow{Tj} TX|_Z \twoheadrightarrow NZ$$

is the identity. Here $0: Z \rightarrow NZ$ denotes the zero section. •

Choose a tubular neighbourhood $j: U \hookrightarrow X$.

Definition 3.7. Set $\hat{U} := [0, 1) \times F$. The **blow-up** of X along Z is the manifold with boundary

$$\hat{X} := \hat{U} \cup_j (X \setminus Z)$$

obtained by gluing \hat{U} and $X \setminus Z$ along j . The **blow-down map** $\beta: \hat{X} \rightarrow X$ is defined by $\beta(r, v) := j(rv)$ for $(r, v) \in \hat{U}$ and $\beta(x) := x$ for $x \in X \setminus Z$. •

Henceforth, identify $U \subset NZ$ and $j(U) \subset X$; moreover, identify $\partial\hat{X} = F$.

Definition 3.8. Set

$$\hat{S} := \beta^*S \quad \text{and} \quad \underline{S} := \hat{S}|_F = \pi^*(S|_Z).$$

Endow $S|_Z$ with the complex structure $I := \gamma(\text{vol}_{NZ})$ and \underline{S} with the quaternionic structure

$$I := \gamma(\text{vol}_{NZ}), \quad J := \gamma(\partial_r), \quad \text{and} \quad K = IJ := \gamma(\partial_\alpha) \in \Gamma(F, \text{End}(\underline{S})).$$

Since $X \setminus Z \hookrightarrow \hat{X}$ is a homotopy-equivalence, I extends uniquely to a Euclidean line bundle

$$\hat{I} \rightarrow \hat{X}.$$

Set

$$\underline{I} := \hat{I}|_F. \quad \bullet$$

3.2 The model operator

The purpose of this subsection is to construct a model \mathring{D} for D near Z . This construction relies on the following.

Definition 3.9 (Restriction of Dirac bundles). Denote the second fundamental form of Z with respect to ∇^T by $\Pi \in \Gamma(Z, \text{Hom}(TZ, \text{Hom}(TZ, NZ)))$. The **restriction** of $(S, \gamma, \nabla, \text{Tor})$ to Z is the quadruple $(S|_Z, \gamma|_{TZ}, \nabla|_Z + \frac{1}{2}\gamma(\Pi), \text{Tor}|_Z)$ with

$$\gamma(\Pi)(v) := \sum_{i=1}^{n-2} \gamma(\Pi(v)e_i)\gamma(e_i).$$

Here (e_1, \dots, e_{n-2}) denotes a local orthonormal frame of TZ . •

Proposition 3.10. $(S|_Z, \gamma|_{TZ}, \nabla|_Z + \frac{1}{2}\gamma(\Pi), \text{Tor}|_Z)$ is a Dirac bundle with skew torsion over $(Z, g|_Z)$.

Proof. Evidently, $(S|_Z, \gamma|_{TZ})$ forms a Clifford module bundle over $(Z, g|_Z)$. Denote by $\nabla^{T, \parallel}$ the orthogonal affine connection on $(Z, g|_Z)$ induces by ∇^T .

Since $(S, \gamma, \nabla, \text{Tor})$ is a Dirac bundle with skew torsion over (X, g) , for every $v, w \in \text{Vect}(Z)$

$$[\nabla_v, \gamma(w)] = \gamma(\nabla_v^{T, \parallel} w) + \gamma(\Pi(v)w);$$

moreover, by direct computation,

$$\begin{aligned} [\gamma(\Pi(v)), \gamma(w)] &= \sum_{i=1}^{n-2} [\gamma(\Pi(v)e_i)\gamma(e_i), \gamma(w)] = \sum_{i=1}^{n-2} \gamma(\Pi(v)e_i)(\gamma(e_i)\gamma(w) + \gamma(w)\gamma(e_i)) \\ &= -2 \sum_{i=1}^{n-2} \gamma(\Pi(v)e_i)\langle e_i, w \rangle = -2\gamma(\Pi(v)w). \end{aligned}$$

A moment's thought shows that if ∇^{\parallel} the Levi-Civita connection of $(Z, g|_Z)$, then

$$\langle \nabla_u^{T, \parallel} v, w \rangle = \langle \nabla_u^{\parallel} v, w \rangle + (\tfrac{1}{2} \text{Tor}|_Z)(u, v, w).$$

This proves the assertion. ■

Proposition 3.11. Denote by $g^{\parallel} := g|_Z$ and g^{\perp} the Euclidean metrics on TZ and NZ induced by g . Denote by $\Pi: U \rightarrow Z$ the projection map and identify $TU = \Pi^*(TZ \oplus NZ)$ using the Levi-Civita connection. Consider $U \subset X$ equipped with the Riemannian metric

$$\mathring{g} := \Pi^*(g^{\parallel} \oplus g^{\perp}).$$

The quadruple $(\mathring{S}, \mathring{\gamma}, \mathring{\nabla}, \mathring{\text{Tor}})$ consisting of

$$\begin{aligned} \mathring{S} &:= \Pi^*(S|_Z), \quad \mathring{\gamma} := \Pi^*(\gamma|_Z), \quad \mathring{\nabla} := \Pi^*(\nabla|_Z + \tfrac{1}{2}\gamma(\Pi)), \\ \text{and } \mathring{\text{Tor}} &:= \Pi^*(\text{Tor}_Z^{3,0} + \text{Tor}_Z^{1,2}) + \text{rpr}_F^*(d\theta \wedge \theta) \end{aligned}$$

is a Dirac bundle with skew torsion over (U, \mathring{g}) . Here $\text{Tor}_Z^{p,q}$ denotes the (p, q) component with respect to $\Lambda^{\bullet}(T^*Z \oplus N^*Z) = \Lambda^{\bullet}T^*Z \otimes \Lambda^{\bullet}N^*Z$ of the restriction of Tor to Z .

Proof. Denote by $\nabla^{T, \parallel}$ and $\nabla^{T, \perp}$ the orthogonal covariant derivatives on TZ and NZ induced by ∇^T respectively. If $v \in \text{Vect}(Z)$ and $w \in \Gamma(Z, NZ)$, then

$$[\nabla_v, \gamma(w)] = \gamma(\nabla_v^{T, \perp} w) - \gamma(\Pi(v)^* w)$$

and, moreover,

$$\begin{aligned} [\gamma(\Pi(v)), \gamma(w)] &= \sum_{i=1}^{n-2} [\gamma(\Pi(v)e_i)\gamma(e_i), \gamma(w)] = - \sum_{i=1}^{n-2} (\gamma(\Pi(v)e_i)\gamma(w) + \gamma(w)\gamma(\Pi(v)e_i))\gamma(e_i) \\ &= 2 \sum_{i=1}^{n-2} \langle \Pi(v)e_i, w \rangle \gamma(e_i) = 2\gamma(\Pi^*(v)w) \end{aligned}$$

This together with the analogous computation in the proof of [Proposition 3.10](#) proves that $\mathring{\gamma}$ is parallel with respect to $\mathring{\nabla}$ and $\Pi^*(\nabla^{T, \parallel} \oplus \nabla^{T, \perp})$. Therefore, it remains to identify the torsion of $\mathring{\nabla}^T := \Pi^*(\nabla^{T, \parallel} \oplus \nabla^{T, \perp})$.

Denote by $\tilde{\cdot}: \text{Vect}(Z) \rightarrow \text{Vect}(U \setminus Z)$ and $\tilde{\cdot}: \Gamma(Z, NZ) \rightarrow \text{Vect}(U \setminus Z)$ the lifting maps. For $u, v \in \text{Vect}(Z)$ and $n, m \in \Gamma(Z, NZ)$, by direct computation,

$$\begin{aligned} \mathring{\nabla}_u^T \tilde{v} - \mathring{\nabla}_v^T \tilde{u} - [\tilde{u}, \tilde{v}] &= \text{Tor}_Z^{3,0}(u, v, -)^\# + \widetilde{[u, v]} - [\tilde{u}, \tilde{v}] \\ &= \text{Tor}_Z^{3,0}(u, v, -)^\# + (\text{pr}_F^* d\theta)(\tilde{u}, \tilde{v}) \otimes \partial_\alpha; \end{aligned}$$

moreover,

$$\mathring{\nabla}_{\tilde{n}}^T \tilde{m} - \mathring{\nabla}_{\tilde{m}}^T \tilde{n} - [\tilde{n}, \tilde{m}] = 0 \quad \text{and} \quad \mathring{\nabla}_{\tilde{v}}^T \tilde{n} - \mathring{\nabla}_{\tilde{n}}^T \tilde{v} - [\tilde{v}, \tilde{n}] = \text{Tor}_Z^{1,2}(v, n, -)^\#.$$

This proves the assertion. \blacksquare

Definition 3.12. Denote by $\mathring{\mathbb{I}}$ the pullback of \mathbb{I} along the projection $U \setminus Z \cong F \times (0, 1) \rightarrow F$. The **model Dirac operator**

$$\mathring{D}: H_{\text{loc}}^1 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}) \rightarrow L_{\text{loc}}^2 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})$$

is the Dirac operator associated with $(\mathring{S}, \mathring{\nabla}, \mathring{\gamma}, \text{Tor})$ twisted by $\mathring{\mathbb{I}}$. \bullet

Remark 3.13. More explicitly, the model Dirac operator \mathring{D} is of the form

$$\mathring{D} = J(\partial_r - r^{-1} I \mathring{\nabla}_{\partial_\alpha}) + D_Z \quad \text{with} \quad D_Z := \sum_{i=1}^{n-2} \mathring{\gamma}(\tilde{e}_i) \mathring{\nabla}_{\tilde{e}_i}$$

with $(\tilde{e}_1, \dots, \tilde{e}_{n-2})$ denoting the horizontal lift of a local g^\parallel -orthonormal frame. \clubsuit

Choose an isometry $\mathring{S} \cong S|_U$ which agrees with $\text{id}_{S|_Z}$ over Z , and an isometry $\mathring{\mathbb{I}} \cong \mathbb{I}|_{U \setminus Z}$; moreover, *henceforth*, regard these as identifications.

Proposition 3.14. *The error term*

$$\text{Err} := D - \mathring{D}: H_{\text{loc}}^1 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}) \rightarrow L_{\text{loc}}^2 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})$$

is of the form

$$\text{Err} = a \mathring{\nabla} + b - \frac{1}{2} \Pi^*(\gamma(H_Z)) + \frac{1}{2} \Pi^*(\gamma(\text{Tor}_Z^{2,1}))$$

with $a \in \Gamma(U, \text{Hom}(T^*U \otimes \mathring{S}, \mathring{S}))$, $b \in \Gamma(U, \text{End}(\mathring{S}))$, and H_Z denoting the mean curvature of Z . Moreover, a and b vanish along Z .

Proof. If (e_1, \dots, e_{n-2}) is a local orthonormal frame of TZ , then

$$\begin{aligned} -\frac{1}{2} \sum_{i=1}^{n-2} \gamma(e_i) \gamma(\Pi)(e_i) &= \frac{1}{2} \sum_{i,j=1}^{n-2} \gamma(e_i) \gamma(e_j) \gamma(\Pi(e_i) e_j) \\ &= -\frac{1}{2} \gamma(H_Z) + \frac{1}{4} \sum_{i,j=1}^{n-2} \gamma(e_i) \gamma(e_j) (\gamma(\Pi(e_i) e_j) - \gamma(\Pi(e_j) e_i)) \\ &= -\frac{1}{2} \gamma(H_Z) + \frac{1}{2} \gamma(\text{Tor}_Z^{2,1}). \end{aligned}$$

Therefore, the assertion follows from the fact that

$$\mathring{g} - g, \quad \mathring{\nabla} - \nabla - \frac{1}{2} \Pi^*(\gamma(\Pi)), \quad \text{and} \quad \mathring{\gamma} - \gamma$$

vanish along Z . \blacksquare

3.3 The model Gelfand–Robbin quotient

By (the proof of) [Proposition 2.8](#), the model minimal extension

$$\mathring{D}_{\min} := \mathring{D} : \text{dom}(\mathring{D}_{\min}) := H_0^1 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}) \rightarrow L^2 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})$$

is closed, densely defined, and symmetric. A moment's thought shows that the domain of the model maximal extension

$$\mathring{D}_{\max} := \mathring{D}_{\min}^*$$

is

$$\text{dom}(\mathring{D}_{\max}) := \{\phi \in H_{\text{loc}}^1 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}) : \phi, \mathring{D}\phi \in L^2 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})\}.$$

The construction from [Section 2.2](#) and [Section 2.3](#) yields the following.

Definition 3.15. The model Gelfand–Robbin quotient is the Hilbert space

$$\mathring{\mathbf{H}} := \frac{\text{dom}(\mathring{D}_{\max})}{\text{dom}(\mathring{D}_{\min})}$$

equipped with the **model Green's form** $\mathring{G} \in \text{Hom}(\Lambda^2 \mathring{\mathbf{H}}, \mathbb{R})$ defined by

$$\mathring{G}([\phi] \wedge [\psi]) := \langle \mathring{D}\phi, \psi \rangle_{L^2} - \langle \phi, \mathring{D}\psi \rangle_{L^2}. \quad \bullet$$

By (the proof of) [Proposition 2.16](#), $(\mathring{\mathbf{H}}, \mathring{G})$ is a symplectic Hilbert space. In the sense of [Remark 2.12](#), $\mathring{\mathbf{H}}$ has contributions from $\{0\} \times F \subset \hat{U}$ and $\{1\} \times F$. Only the former is relevant for the purposes of this section.

Proposition 3.16. *The subspace*

$$\mathring{\mathbf{H}}_0 := \frac{\chi(r) \cdot \text{dom}(\mathring{D}_{\max}) + \text{dom}(\mathring{D}_{\min})}{\text{dom}(\mathring{D}_{\min})} \subset \mathring{\mathbf{H}}$$

is closed and symplectic.

Proof. Define the operator $\pi \in \mathcal{L}(\mathring{\mathbf{H}})$ by $\pi([\phi]) := [\chi(r) \cdot \phi]$. Since $\chi(r)(1 - \chi(r)) \cdot \text{dom}(\mathring{D}_{\max}) \subset \text{dom}(\mathring{D}_{\min})$, $\pi^2 = \pi$; that is: π is a projection. Hence, $\mathring{\mathbf{H}}_0 = \text{im } \pi = \ker(1 - \pi)$ is closed.

Since $(1 - \chi(r) - \chi \circ (1 - r)) \cdot \text{dom}(\mathring{D}_{\max}) \subset \text{dom}(\mathring{D}_{\min})$, $(1 - \pi)[\phi] = [\chi \circ (1 - r) \cdot \phi]$. Therefore,

$$\mathring{G}([\phi] \wedge [\psi]) = \mathring{G}(\pi[\phi] \wedge \pi[\psi]) + \mathring{G}((1 - \pi)[\phi] \wedge (1 - \pi)[\psi]).$$

Hence, $\mathring{\mathbf{H}}_0$ is symplectic. ■

Proposition 3.17. *There is a unique isomorphism of symplectic Hilbert spaces*

$$\text{cut-off}: (\check{\mathbf{H}}, G) \cong (\mathring{\mathbf{H}}_0, \mathring{G})$$

satisfying $\text{cut-off}([\phi]) = [\chi(r) \cdot \phi]$ *for every* $\phi \in \text{dom}(\mathring{D}_{\max})$.

The proof requires the following preparation.

Lemma 3.18 ($\text{dom}(D_{\max})$ vs. $\text{dom}(\dot{D}_{\max})$). *The following hold:*

- (1) If $\phi \in \text{dom}(\dot{D}_{\max})$, then $\chi(r) \cdot \phi \in \text{dom}(D_{\max})$ and $\|\chi(r) \cdot \phi\|_D \lesssim \|\phi\|_{\dot{D}}$.
- (2) If $\phi \in \text{dom}(D_{\max})$, then $\chi(r) \cdot \phi \in \text{dom}(\dot{D}_{\max})$ and $\|\chi(r) \cdot \phi\|_{\dot{D}} \lesssim \|\phi\|_D$.

Proof. Let $\phi \in \text{dom}(\dot{D}_{\max})$. Let $\eta \in C_c^\infty(U \setminus Z, [0, 1])$. By [Proposition 2.5 \(2\)](#),

$$\begin{aligned} \int_{U \setminus Z} \eta^2 |\dot{\nabla}(r\chi(r) \cdot \phi)|^2 &= \int_{U \setminus Z} \eta^2 |\dot{D}(r\chi(r) \cdot \phi)|^2 \\ &\quad - \int_{U \setminus Z} \eta^2 (\langle \tau \dot{\nabla}(r\chi(r) \cdot \phi), r\chi(r) \cdot \phi \rangle + \langle \dot{\gamma}(F_{\dot{\nabla}})r\chi(r) \cdot \phi, r\chi(r) \cdot \phi \rangle) \\ &\quad + 2 \int_{U \setminus Z} \eta \langle \dot{D}(r\chi(r) \cdot \phi), \dot{\gamma}(d\eta)r\chi(r) \cdot \phi \rangle \\ &\quad - 2 \int_{U \setminus Z} \eta \langle \dot{\nabla}(r\chi(r) \cdot \phi), d\eta \otimes r\chi(r) \cdot \phi \rangle. \end{aligned}$$

Therefore,

$$\int_{U \setminus Z} \eta^2 |\dot{\nabla}(r\chi(r) \cdot \phi)|^2 \lesssim \int_{U \setminus Z} |\dot{D}\phi|^2 + r^2(1 + |d\eta|^2)|\phi|^2.$$

Since $\eta_\varepsilon := 1 - \chi(r/\varepsilon)$ satisfies $r|d\eta_\varepsilon| \lesssim 1$,

$$\int_{U \setminus Z} |\dot{\nabla}(r\chi(r) \cdot \phi)|^2 = \lim_{\varepsilon \downarrow 0} \int_{U \setminus Z} \eta_\varepsilon^2 |\dot{\nabla}(r\phi)|^2 \lesssim \int_{U \setminus Z} |\dot{D}\phi|^2 + |\phi|^2.$$

Therefore, $r\chi(r) \cdot \phi \in \text{dom}(\dot{D}_{\min})$ and

$$\|r\chi(r) \cdot \phi\|_{H^1} \lesssim \|\phi\|_{\dot{D}}.$$

By [Proposition 3.14](#) and the above,

$$\|\text{Err } \chi(r) \cdot \phi\|_{L^2} \lesssim \|\dot{\nabla}(r\phi)\|_{L^2} + \|\phi\|_{L^2} \lesssim \|\dot{D}\phi\|_{L^2} + \|\phi\|_{L^2}.$$

This implies (1). The proof of (2) is similar. ■

Proof of Proposition 3.17. By [Lemma 3.18](#), cut-off is an isomorphism of Hilbert spaces. To prove that cut-off is a symplectomorphism, let $\phi, \psi \in \text{dom}(D_{\max})$ and set

$$v := \sum_{i=1}^n \langle \gamma(e_i)\phi, \psi \rangle e_i \quad \text{and} \quad \dot{v} := \sum_{i=1}^n \langle \dot{\gamma}(\dot{e}_i)\chi(r) \cdot \phi, \chi(r) \cdot \psi \rangle \dot{e}_i$$

with (e_1, \dots, e_n) and $(\dot{e}_1, \dots, \dot{e}_n)$ denoting local g - and \dot{g} -orthonormal frames respectively.

Assume, without loss of generality, that $\text{supp}(\phi) \cup \text{supp}(\psi) \subset (\chi \circ r)^{-1}(1) \subset U \setminus Z$. With η_ε as in the proof of [Lemma 3.18](#),

$$\begin{aligned} (G - \text{cut-off}^* \mathring{G})([\phi] \wedge [\psi]) &= \int_{U \setminus Z} \text{div}_g(v) \cdot \text{vol}_g - \text{div}_{\mathring{g}}(\mathring{v}) \cdot \text{vol}_{\mathring{g}} \\ &= \lim_{\varepsilon \downarrow 0} \int_{U \setminus Z} \eta_\varepsilon \cdot (\text{div}_g(v) \cdot \text{vol}_g - \text{div}_{\mathring{g}}(\mathring{v}) \cdot \text{vol}_{\mathring{g}}) \\ &= - \lim_{\varepsilon \downarrow 0} \int_{U \setminus Z} d\eta_\varepsilon \wedge (i_v \text{vol}_g - i_{\mathring{v}} \text{vol}_{\mathring{g}}). \end{aligned}$$

Since $r|d\eta_\varepsilon| \lesssim 1$,

$$|d\eta_\varepsilon \wedge (i_v \text{vol}_g - i_{\mathring{v}} \text{vol}_{\mathring{g}})| \lesssim |\phi| |\psi|.$$

Therefore,

$$|(G - \text{cut-off}^* \mathring{G})([\phi] \wedge [\psi])| \lesssim \lim_{\varepsilon \downarrow 0} \int_{\text{supp } d\eta_\varepsilon} |\phi| |\psi| \text{vol}_{\mathring{g}} = 0. \quad \blacksquare$$

3.4 Spectral decomposition

This subsection decomposes $(\mathring{\mathbf{H}}_0, \mathring{G})$ into concretely understandable summands.

Definition 3.19 (\mathbf{I} determines NZ^λ). The ramified Euclidean line bundle \mathbf{I} determines the following:

- (1) The 2π -periodic vector field ∂_α generating $F \curvearrowright \text{U}(1)$ uniquely lifts along

$$\rho: \tilde{F} := \{\ell \in \mathbb{I} : |\ell| = 1\} \rightarrow F$$

to a 4π -periodic vector field $\frac{1}{2}\partial_\beta$. The 2π -periodic vector field ∂_β generates $\tilde{F} \curvearrowright \text{U}(1)$ with respect to which $\tilde{\pi}: \tilde{F} \rightarrow Z$ is a $\text{U}(1)$ -principal bundle.

- (2) Let $\lambda \in \frac{1}{2}\mathbb{Z}$. The Hermitian line bundle

$$NZ^\lambda := \tilde{F} \times_{\text{U}(1)} \mathbb{C}$$

arises from \tilde{F} via the representation $\text{U}(1) \curvearrowright \mathbb{C}$ of weight 2λ . The Levi-Civita connection on F induces a connection on \tilde{F} and, therefore, a unitary covariant derivative ∇^λ on NZ^λ . •

Remark 3.20. By construction $(NZ^1, \nabla^1) \cong (NZ, \nabla^{\text{LC}})$ and for every $\lambda, \mu \in \frac{1}{2}\mathbb{Z}$

$$(NZ^\lambda, \nabla^\lambda) \otimes_{\mathbb{C}} (NZ^\mu, \nabla^\mu) \cong (NZ^{\lambda+\mu}, \nabla^{\lambda+\mu}). \quad \clubsuit$$

Proposition 3.21. For every $\lambda \in \mathbb{Z} - 1/2$ there is an isomorphism

$$P_\lambda: \pi^*(NZ^\lambda, \nabla^\lambda) \cong (\mathbb{I} \otimes \mathbb{C}, \nabla_{\mathbb{I} \otimes \mathbb{C}} + i\lambda\theta)$$

of Hermitian line bundles with unitary connections.

Proof. Consider the $U(1)$ -principal bundle $\tilde{F} \times_{\{\pm 1\}} U(1) \rightarrow F$ obtained by extending the $\{\pm 1\}$ -principal bundle $\tilde{F} \rightarrow F$ along the inclusion $\iota: \{\pm 1\} \hookrightarrow U(1)$. The $U(1)$ -principal bundles $\tilde{F} \times_{\{\pm 1\}} U(1) \rightarrow F$ and $\pi^* \tilde{F} \rightarrow F$ are isomorphic via $[f, z] \mapsto [f, \rho(f \cdot z)]$.

Let $\lambda \in \mathbb{Z} - 1/2$. The representation $U(1) \cup \mathbb{C}$ of weight 2λ restricts to the usual representation $\{\pm 1\} \cup \mathbb{C}$ along ι . Therefore, $\pi^* NZ^\lambda$ and \underline{I} both arise from the representation of weight 2λ . Hence, they are isomorphic as Hermitian line bundles.

The Levi-Civita connection $i\theta$ on $F \rightarrow Z$ induces the connection $\frac{i}{2}\rho^*\theta$ on $\tilde{F} \rightarrow Z$. Therefore, the induced connection on $\tilde{F} \times_{\{\pm 1\}} U(1) \cong \pi^* F$ is (the descend of) $\frac{i}{2}\rho^*\theta + \mu_{U(1)}$. Here $\mu_{U(1)} \in U(1)$ denotes the Maurer–Cartan form on $U(1)$. The connection on $\tilde{F} \times_{\{\pm 1\}} U(1)$ induced by the flat connection on $\tilde{F} \rightarrow F$ is (the descend) of $\mu_{U(1)}$. This implies the assertion about the covariant derivatives. \blacksquare

Proposition 3.22 (Spectral decomposition of $L^2\Gamma(F, \underline{S} \otimes \underline{I})$). *For every $\lambda \in \mathbb{Z} - 1/2$ and $\mu \in \mathbb{R}$ set*

$$\begin{aligned} E_{\lambda, \mu} &:= \{\phi \in L^2\Gamma(F, \underline{S} \otimes \underline{I}) : I\mathring{\nabla}_{\partial_\alpha} \phi = \lambda \phi, D_Z \phi = \mu \phi\} \quad \text{and} \\ \check{E}_{\lambda, \mu} &:= \{\check{\phi} \in L^2\Gamma(Z, S|_Z \otimes_{\mathbb{C}} NZ^\lambda) : D_{S|_Z \otimes_{\mathbb{C}} NZ^\lambda} \check{\phi} = \mu \check{\phi}\}. \end{aligned}$$

Here D_Z is as in [Remark 3.13](#) and $D_{S|_Z \otimes_{\mathbb{C}} NZ^\lambda}$ arises from [Definition 3.9](#) and twisting by $(NZ^\lambda, \nabla^\lambda)$; moreover: the tensor product is with respect to the complex structure I on $S|_Z$. The following hold:

- (1) For every $\lambda \in \mathbb{Z} - 1/2$ and $\mu \in \mathbb{R}$, P_λ induces an isomorphism

$$\pi^* \check{E}_{\lambda, \mu} \cong E_{\lambda, \mu}.$$

- (2) The subset

$$\sigma := \{(\lambda, \mu) \in (\mathbb{Z} - 1/2) \times \mathbb{R} : E_{\lambda, \mu} \neq 0\}$$

is discrete. Moreover, for every $(\lambda, \mu) \in \sigma$, $E_{\lambda, \mu}$ is finite-dimensional.

- (3) For every $(\lambda, \mu) \in \sigma$

$$JE_{\lambda, \mu} = E_{-(\lambda+1), -\mu}.$$

- (4) The Hilbert space $L^2\Gamma(F, \underline{S} \otimes \underline{I})$ decomposes as a (Hilbert space) direct sum

$$L^2\Gamma(F, \underline{S} \otimes \underline{I}) = \bigoplus_{(\lambda, \mu) \in \sigma} E_{\lambda, \mu}.$$

Proof. By Fourier analysis, the Hilbert space $L^2\Gamma(F, \underline{S} \otimes \underline{I})$ decomposes as a direct sum

$$L^2\Gamma(F, \underline{S} \otimes \underline{I}) = \bigoplus_{\lambda \in \mathbb{Z} - 1/2} E_\lambda \quad \text{with} \quad E_\lambda := \{\phi \in L^2\Gamma(F, \underline{S} \otimes \underline{I}) : I\mathring{\nabla}_{\partial_\alpha} \phi = \lambda \phi\}$$

By [Proposition 3.21](#), P_λ induces an isomorphism $\pi^* L^2\Gamma(Z, S|_Z \otimes_{\mathbb{C}} NZ^\lambda) \cong E_\lambda$. By the spectral theory of Dirac operators, for every $\lambda \in \mathbb{Z} - 1/2$, $\text{spec}(D_{S|_Z \otimes_{\mathbb{C}} NZ^\lambda}) \subset \mathbb{R}$ is discrete and the Hilbert space $L^2\Gamma(Z, S|_Z \otimes_{\mathbb{C}} NZ^\lambda)$ decomposes as a direct sum finite-dimensional eigenspaces $\check{E}_{\lambda, \mu}$ of $D_{S|_Z \otimes_{\mathbb{C}} NZ^\lambda}$. This proves (1), (2), and (4).

- (3) holds because J and D_Z anti-commute and $I\mathring{\nabla}_{\partial_\alpha} J = -J(I\mathring{\nabla}_{\partial_\alpha} + 1)$ since $\mathring{\nabla}_{\partial_\alpha} \partial_r = \partial_\alpha$. \blacksquare

Since $\text{vol}_{\check{g}} = dr \wedge r\theta \wedge \text{vol}_{g|_Z}$, by Fubini's theorem and [Proposition 3.22](#),

$$L^2\Gamma(U \setminus Z, \check{S} \otimes \check{\mathbb{I}}) = L^2((0, 1), r dr; L^2\Gamma(F, \underline{S} \otimes \underline{\mathbb{I}})) = \bigoplus_{\lambda \in \mathbb{Z} - 1/2} \bigoplus_{\mu \in \sigma_\lambda} L^2((0, 1), r dr; E_{\lambda, \mu});$$

moreover, D_{\max} decomposes as follows.

Definition 3.23. Choose a fundamental domain $\check{\sigma} \subset \sigma$ for the involution $(\lambda, \mu) \mapsto -(\lambda + 1, -\mu)$. Choose a real subspace $E_{-1/2, 0}^{\mathbb{R}} \subset E_{-1/2, 0}$ with respect to J . For every $(\lambda, \mu) \in \check{\sigma}$ set

$$V_{\lambda, \mu} := \begin{cases} E_{-1/2, 0}^{\mathbb{R}} \oplus J E_{-1/2, 0}^{\mathbb{R}} & \text{if } (\lambda, \mu) = (-1/2, 0) \\ E_{\lambda, \mu} \oplus E_{-(\lambda+1), -\mu} & \text{otherwise;} \end{cases}$$

moreover, define $\check{D}^{\lambda, \mu}: H_{\text{loc}}^1((0, 1); V_{\lambda, \mu}) \rightarrow L_{\text{loc}}^2((0, 1); V_{\lambda, \mu})$ by

$$\check{D}^{\lambda, \mu} := \begin{pmatrix} \mu & J(\partial_r + \frac{\lambda+1}{r}) \\ J(\partial_r - \frac{\lambda}{r}) & -\mu \end{pmatrix}$$

and set

$$\text{dom}(\check{D}_{\max}^{\lambda, \mu}) := \left\{ \phi \in H_{\text{loc}}^1((0, 1); V_{\lambda, \mu}) : \phi, \check{D}\phi \in L^2((0, 1), r dr; V_{\lambda, \mu}) \right\}. \quad \bullet$$

Remark 3.24. The purpose of the artificial decomposition of $E_{-1/2, 0}$ is to avoid a case distinction in the definition of $\check{D}^{\lambda, \mu}$. ♣

Proposition 3.25 (Spectral decomposition of $\text{dom}(\check{D}_{\max})$). *The following hold:*

- (1) *The Hilbert space $\text{dom}(\check{D}_{\max})$ decomposes as a (Hilbert space) direct sum*

$$\text{dom}(\check{D}_{\max}) = \bigoplus_{(\lambda, \mu) \in \check{\sigma}} \text{dom}(\check{D}_{\max}^{\lambda, \mu}).$$

- (2) *The model operator \check{D} decomposes as*

$$\check{D} = \bigoplus_{(\lambda, \mu) \in \check{\sigma}} \check{D}^{\lambda, \mu}.$$

Proof. This is an immediate consequence of [Remark 3.13](#) and [Proposition 3.22](#). ■

Remark 3.26. The ordinary differential equation $\check{D}^{\lambda, \mu}\phi = \psi$ can be solved explicitly in terms of modified Bessel functions of the second kind or using the Hankel transform. However, none of this is necessary for the purpose of this article. ♣

Finally, here is the desired decomposition of $(\check{\mathbf{H}}_0, \check{G})$.

Corollary 3.27 (Spectral decomposition of $(\check{\mathbf{H}}_0, \check{G})$). *The symplectic Hilbert space $(\check{\mathbf{H}}_0, \check{G})$ decomposes as a (Hilbert space) direct sum*

$$(\check{\mathbf{H}}_0, \check{G}) = \bigoplus_{(\lambda, \mu) \in \check{\sigma}} (\check{\mathbf{H}}_0^{\lambda, \mu}, \check{G}_{\lambda, \mu})$$

with

$$\begin{aligned}\mathring{H}_0^{\lambda,\mu} &:= \frac{(\chi \circ r) \cdot \text{dom}(\mathring{D}_{\max}^{\lambda,\mu}) + \text{dom}(\mathring{D}_{\min})}{\text{dom}(\mathring{D}_{\min})} \quad \text{and} \\ \mathring{G}_{\lambda,\mu}([\phi] \wedge [\psi]) &:= \langle \mathring{D}^{\lambda,\mu} \phi, \psi \rangle_{L^2} - \langle \phi, \mathring{D}^{\lambda,\mu} \psi \rangle_{L^2}.\end{aligned}$$

■

3.5 Leading order terms

This subsection determines $(\mathring{H}_0^{\lambda,\mu}, \mathring{G}_{\lambda,\mu})$ based on the following observation.

Lemma 3.28 (Leading order terms; cf. [BS88, Lemma 2.1; DW24, Lemma 3.50]). *Let $\lambda \in \mathbf{R}$. Let $\phi \in H_{\text{loc}}^1((0, 1))$ with $\phi, (\partial_r - \lambda/r)\phi \in L^2((0, 1), r dr)$. The following hold:*

- (1) *If $\lambda \in (-1, 0)$, then there is a unique $a \in \mathbf{R}$ such that $\lim_{r \downarrow 0} \phi(r) - ar^\lambda = 0$.*
- (2) *If $\lambda = 0$, then $\phi(r) \lesssim_\phi |\log(r)|^{1/2}$.*
- (3) *If $\lambda \notin [-1, 0)$, then $\lim_{r \downarrow 0} \phi(r) = 0$.*
- (4) *If $\lim_{r \downarrow 0} \phi(r) = \lim_{r \uparrow 1} \phi(r) = 0$, then*

$$\int_0^1 \left(|\partial_r \phi|^2 + \frac{\lambda^2}{r^2} |\phi|^2 \right) r dr = \int_0^1 |(\partial_r - \lambda/r)\phi|^2 r dr.$$

Proof. The proof is almost identical to that of [DW24, Lemma 3.50] and is repeated here only for the readers' convenience.

Evidently, $(\partial_r - \lambda/r)r^\lambda = 0$ and $r^\lambda \in L^2((0, 1), r dr)$ if and only if $\lambda > -1$. Set $\psi := (\partial_r - \lambda/r)\phi$. By variation of parameters, there is a unique $a \in \mathbf{R}$ such that

$$\tilde{\phi}(r) := \phi(r) - ar^\lambda = \begin{cases} r^\lambda \int_0^r s^{-(\lambda+1)} \psi(s) s ds & \text{if } \lambda < 0 \\ -r^\lambda \int_r^1 s^{-(\lambda+1)} \psi(s) s ds & \text{if } \lambda \geq 0. \end{cases}$$

Of course, if $\lambda \leq -1$, then $a = 0$.

If $\lambda < 0$, then, by Cauchy–Schwarz and monotone convergence,

$$|\tilde{\phi}(r)|^2 \leq \frac{1}{2|\lambda|} \int_0^r |\psi(s)|^2 s ds = o(1) \quad \text{as } r \downarrow 0.$$

If $\lambda = 0$, then

$$|\tilde{\phi}(r)|^2 \leq |\log(r)| \int_r^1 |\psi(s)|^2 s ds = O(|\log(r)|) \quad \text{as } r \downarrow 0.$$

If $\lambda > 0$, then, by Cauchy–Schwarz, for $r \leq \varepsilon \leq 1$

$$|\tilde{\phi}(r)|^2 \leq \frac{1}{\lambda} \int_0^\varepsilon |\psi(s)|^2 s ds + \frac{(r/\varepsilon)^{2\lambda}}{\lambda} \int_\varepsilon^1 |\psi(s)|^2 s ds =: I(\varepsilon) + \Pi(r, \varepsilon).$$

By monotone convergence, $\lim_{\varepsilon \downarrow 0} I(\varepsilon) = 0$. Evidently, $\lim_{r \downarrow 0} \Pi(r, \varepsilon) = 0$. Therefore, $\tilde{\phi}(r) = o(1)$ as $r \downarrow 0$. These observations imply (1), (2), and (3).

(4) is a consequence of

$$\int_0^1 |(\partial_r - \lambda/r)\phi|^2 r dr = \int_0^1 \left(|\partial_r \phi|^2 + \frac{\lambda^2}{r^2} |\phi|^2 \right) r dr - \lambda \int_0^1 \partial_r |\phi|^2 dr. \quad \blacksquare$$

Corollary 3.29 (Identification of $\mathring{H}_0^{\lambda, \mu}$ for $\lambda \neq -1/2$). *For every $(\lambda, \mu) \in \check{\sigma}$ with $\lambda \neq -1/2$*

$$\mathring{H}_0^{\lambda, \mu} = 0. \quad \blacksquare$$

Definition 3.30. For $(-1/2, \mu) \in \check{\sigma}$ define the **residue map** $\text{res}_\mu : \text{dom}(\mathring{D}_{\max}^{-1/2, \mu}) \rightarrow V_{-1/2, \mu}$ by

$$\phi - r^{-1/2} \cdot \text{res}_\mu(\phi) \in \text{dom}(\mathring{D}_{\min}),$$

and the symplectic form $\check{\Omega}_\mu \in \text{Hom}(\Lambda^2 V_{-1/2, \mu}, \mathbf{R})$ by

$$\check{\Omega}_\mu(\phi \wedge \psi) := -\langle J\phi, \psi \rangle. \quad \bullet$$

Proposition 3.31 (Identification of $\mathring{H}_0^{-1/2, \mu}$: symplectic structure). *For every $(-1/2, \mu) \in \check{\sigma}$ the residue map induces an isomorphism*

$$\text{res}_\mu : (\mathring{H}_0^{-1/2, \mu}, \mathring{G}^{-1/2, \mu}) \cong (V_{-1/2, \mu}, \check{\Omega}_\mu).$$

Proof. For $\phi, \psi \in \text{dom}(\mathring{D}_{\max}^{-1/2, \mu})$, by direct computation using $\partial_r + \frac{1}{2r} = r^{-1/2} \partial_r r^{1/2}$,

$$\begin{aligned} \mathring{G}^{-1/2, \mu}([\phi] \wedge [\psi]) &= \int_0^1 (\langle J(\partial_r + \frac{1}{2r})\phi, \psi \rangle - \langle \phi, J(\partial_r + \frac{1}{2r})\psi \rangle) r dr \\ &= \int_0^1 \partial_r \langle J r^{1/2} \phi, r^{1/2} \psi \rangle dr \\ &= -\langle J \text{res}_\mu([\phi]), \text{res}_\mu([\psi]) \rangle = \text{res}_\mu^* \check{\Omega}_\mu([\phi] \wedge [\psi]). \end{aligned}$$

This together with [Lemma 3.28](#) immediately implies the assertion. \blacksquare

Although res_μ is an isomorphism, the norms on $\mathring{H}_0^{-1/2, \mu}$ and $V_{-1/2, \mu}$ are *not uniformly* equivalent. The following discussion rectifies this.

Definition 3.32. Let $(-1/2, \mu) \in \check{\sigma}$.

(1) Define the **branching locus operator** $A_\mu : V_{-1/2, \mu} \rightarrow V_{-1/2, \mu}$ by

$$A_\mu := -J \mathring{D}^{-1/2, \mu} - \partial_r - \frac{1}{2r} = \begin{pmatrix} 0 & J\mu \\ -J\mu & 0 \end{pmatrix}.$$

(2) Define the norm $\|-\|_{\check{H}} : V_{-1/2, \mu} \rightarrow [0, \infty)$ by

$$\|v\|_{\check{H}}^2 := (1 + |\mu|) \cdot \|\mathbf{1}_{(-\infty, 0)}(A_\mu)\phi\|^2 + (1 + |\mu|)^{-1} \cdot \|\mathbf{1}_{[0, \infty)}(A_\mu)\phi\|^2.$$

Here $\mathbf{1}_{(-\infty, 0)}(A_\mu)$ and $\mathbf{1}_{[0, \infty)}(A_\mu)$ denote the orthogonal projection to the negative and non-negative eigenspaces of A_μ respectively.

(3) Define the norm $\| \cdot \|_{H^{-1/2}} : V_{-1/2, \mu} \rightarrow [0, \infty)$ by

$$\|v\|_{H^{-1/2}}^2 := (1 + |\mu|)^{-1} \cdot \|v\|^2. \quad \bullet$$

Proposition 3.33 (Identification of $\mathring{\mathbf{H}}_0^{-1/2, \mu}$: uniform norms). *For every $(-1/2, \mu) \in \check{\sigma}$*

$$\|\text{res}_\mu([\phi])\|_{\check{H}} \asymp \|[\phi]\|_{\mathring{\mathbf{H}}}.$$

The proof uses the following right inverse of $\text{res}_\mu \circ [\cdot]$.

Definition 3.34. For $(-1/2, \mu) \in \check{\sigma}$ define the **extension map** $\text{ext}_\mu : V_{-1/2, \mu} \rightarrow \text{dom}(\mathring{D}_{\max}^{-1/2, \mu})$ by

$$\text{ext}_\mu(v) := r^{-1/2} e^{-|\mu|r} \cdot v. \quad \bullet$$

Evidently, ext_μ lifts the inverse of $\text{res}_\mu : \mathring{\mathbf{H}}_0^{-1/2, \mu} \cong V_{-1/2, \mu}$. Therefore, [Proposition 3.33](#) is an immediate consequence of the following.

Lemma 3.35 (Uniform estimates for res_μ and ext_μ). *Let $(-1/2, \mu) \in \check{\sigma}$. The following hold:*

(1) *For every $\phi \in \text{dom}(\mathring{D}_{\max}^{-1/2, \mu})$*

$$\|\text{res}_\mu([\phi])\|_{\check{H}} \lesssim \|\phi\|_{\mathring{D}}.$$

(2) *For every $v \in V_{-1/2, \mu}$*

$$\|\text{ext}_\mu(v)\|_{L^2} \lesssim \|v\|_{H^{-1/2}} \quad \text{and} \quad \|\text{ext}_\mu(v)\|_{\mathring{D}} \lesssim \|v\|_{\check{H}}.$$

Proof. Evidently, $v := \text{res}_\mu([\phi])$ satisfies

$$v = - \int_0^1 \partial_r (r^{1/2} e^{-|\mu|r} \phi) \, dr = - \int_0^1 r^{1/2} (-J \mathring{D}^{-1/2, \mu} - A_\mu - |\mu|) \phi \cdot e^{-|\mu|r} \, dr.$$

Therefore, by Cauchy–Schwarz,

$$\begin{aligned} \|\mathbf{1}_{(-\infty, 0)}(A_\mu) v\|^2 &\lesssim \int_0^1 \|\mathring{D}^{-1/2, \mu} \mathbf{1}_{(-\infty, 0)}(A_\mu) \phi\|^2 r \, dr \cdot \int_0^1 e^{-2|\mu|r} \, dr \\ &\lesssim \int_0^1 e^{-2|\mu|r} \, dr \cdot \|\mathbf{1}_{(-\infty, 0)}(A_\mu) \phi\|_{\mathring{D}}^2 \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{1}_{[0, \infty)}(A_\mu) v\|^2 &\lesssim \int_0^1 \left(\|\mathring{D}^{-1/2, \mu} \mathbf{1}_{[0, \infty)}(A_\mu) \phi\|^2 + |\mu|^2 \|\mathbf{1}_{[0, \infty)}(A_\mu) \phi\|^2 \right) r \, dr \cdot \int_0^1 e^{-2|\mu|r} \, dr \\ &\lesssim (1 + |\mu|)^2 \cdot \int_0^1 e^{-2|\mu|r} \, dr \cdot \|\mathbf{1}_{[0, \infty)}(A_\mu) \phi\|_{\mathring{D}}^2. \end{aligned}$$

The estimate in [\(1\)](#) follows because

$$\int_0^1 e^{-2|\mu|r} \, dr \asymp (1 + |\mu|)^{-1} \quad \text{and} \quad \|\mathbf{1}_{(-\infty, 0)}(A_\mu) \phi\|_{\mathring{D}}^2 + \|\mathbf{1}_{[0, \infty)}(A_\mu) \phi\|_{\mathring{D}}^2 = \|\phi\|_{\mathring{D}}^2.$$

To prove (2), observe that

$$\|\text{ext}_\mu(v)\|_{L^2}^2 \lesssim \int_0^1 r^{-1} e^{-2|\mu|r} \|v\|^2 r dr \lesssim (1 + |\mu|)^{-1} \cdot \|v\|^2$$

and

$$\begin{aligned} \|\mathring{D}^{-1/2, \mu} \text{ext}_\mu(v)\|_{L^2}^2 &= \int_0^1 (r^{-1/2} e^{-|\mu|r})^2 \|(-|\mu| + A_\mu)v\|^2 r dr \\ &\lesssim \int_0^1 (r^{-1/2} |\mu| e^{-|\mu|r})^2 \|\mathbf{1}_{(-\infty, 0]}(A_\mu)v\|^2 r dr \\ &\lesssim \mu^2 (1 + |\mu|)^{-1} \cdot \|v\|^2. \end{aligned} \quad \blacksquare$$

3.6 Assembly of the residue map

This subsection (re)assembles the summands of the decomposition [Corollary 3.27](#) identified in [Section 3.5](#) in a more geometric fashion.

Definition 3.36.

- (1) The **residue bundle** is

$$\check{S} := S|_Z \otimes_{\mathbb{C}} NZ^{-1/2}.$$

As a consequence of [Proposition 3.22 \(3\)](#) (or by direct inspection), \check{S} inherits the quaternionic structure $I, J, K = IJ$ from \underline{S} . Define the symplectic form $\check{\Omega} \in \Gamma(Z, \text{Hom}(\Lambda^2 \check{S}, \mathbb{R}))$ by

$$\check{\Omega} := -2\pi \langle J-, - \rangle.$$

- (2) The **branching locus operator** $A: \Gamma(Z, \check{S}) \rightarrow \Gamma(Z, \check{S})$ is defined by

$$A := -JD_{\check{S}}$$

with $D_{\check{S}} := D_{S|_Z \otimes_{\mathbb{C}} NZ^{-1/2}}$ as in [Proposition 3.22](#). Since J and $D_{\check{S}}$ anti-commute, A is (formally) self-adjoint.

- (3) Denote by $\mathbf{1}_{(-\infty, 0)}(A)$ and $\mathbf{1}_{[0, \infty)}(A)$ the orthogonal projection to the negative and non-negative eigenspaces of A respectively. Define the norm $\|-\|_{\check{H}}: \Gamma(Z, \check{S}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{\check{H}} := \|\mathbf{1}_{(-\infty, 0)}(A)\phi\|_{H^{1/2}} + \|\mathbf{1}_{[0, \infty)}(A)\phi\|_{H^{-1/2}}$$

and denote by $\check{H}\Gamma(Z, \check{S})$ the completions of $\Gamma(Z, \check{S})$ with respect to $\|-\|_{\check{H}}$. •

Proposition 3.37. $\check{\Omega}$ extends to a symplectic structure $\check{\Omega} \in \mathcal{L}(\Lambda^2 \check{H}\Gamma(Z, \check{S}), \mathbb{R})$; moreover: the inclusion $V_{-1/2, \mu} \hookrightarrow \Gamma(Z, \check{S})$ assemble into an isomorphism of symplectic Hilbert spaces

$$\bigoplus_{(-1/2, \mu) \in \check{\sigma}} (V_{-1/2, \mu}, \|-\|_{\check{H}}; \check{\Omega}_\mu) \cong (\check{H}\Gamma(Z, \check{S}); \check{\Omega}).$$

Proof. This is an immediate consequence of [Proposition 3.22](#), [Proposition 3.31](#), and [Proposition 3.33](#). The (possibly mysterious) factor 2π arises because $\check{\Omega}_\mu$ is defined using the L^2 inner product on F instead of Z and $\text{vol}(F) = 2\pi\text{vol}(Z)$. \blacksquare

Definition 3.38.

- (1) The **residue map** $\text{res}: \check{H} \rightarrow \check{H}\Gamma(Z, \check{S})$ obtained as the composition of the following maps

$$\check{H} \xrightarrow{\text{cut-off}} \check{H}_0 = \bigoplus_{(-1/2, \mu) \in \check{\sigma}} \check{H}_0^{-1/2, \mu} \xrightarrow{(\text{res}_\mu)} \bigoplus_{(-1/2, \mu) \in \check{\sigma}} (V_{-1/2, \mu}, \|\cdot\|_{\check{H}}) \cong \check{H}\Gamma(Z, \check{S}).$$

- (2) The **extension map** $\text{ext}: \check{H}\Gamma(Z, \check{S}) \rightarrow \text{dom}(D_{\max})$ is obtained as the composition of the following maps

$$\begin{aligned} \check{H}\Gamma(Z, \check{S}) &\cong \bigoplus_{(-1/2, \mu) \in \check{\sigma}} (V_{-1/2, \mu}, \|\cdot\|_{\check{H}}) \xrightarrow{(\text{ext}_\mu)} \bigoplus_{(-1/2, \mu) \in \check{\sigma}} \text{dom}(\check{D}_{\max}^{-1/2, \mu}) \\ &\hookrightarrow \text{dom}(\check{D}_{\max}) \xrightarrow{\chi^{\text{or}}} \text{dom}(D_{\max}). \quad \bullet \end{aligned}$$

Theorem 3.39. *The following hold:*

- (1) (a) *The residue map is an isomorphism of symplectic Hilbert spaces*

$$\text{res}: (\check{H}, G) \cong (\check{H}\Gamma(Z, \check{S}), \check{\Omega}).$$

- (b) *The subspace $r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{I}) \cap \text{dom}(D_{\max})$ is dense in $\text{dom}(D_{\max})$; hence: the residue map is uniquely determined by*

$$\pi^* \text{res}[r^{-1/2}\phi] = \phi|_{\partial\hat{X}}$$

for every $r^{-1/2}\phi \in r^{-1/2}\Gamma(\hat{X}, \hat{S} \otimes \hat{I}) \cap \text{dom}(D_{\max})$.

- (2) (a) *The extension map $\text{ext}: \check{H}\Gamma(Z, \check{S}) \rightarrow \text{dom}(D_{\max})$ is a right-inverse of $\text{res} \circ [-]$.*
(b) *The extension map extends to a bounded linear map*

$$\text{ext}: H^{-1/2}\Gamma(Z, \check{S}) \rightarrow L^2\Gamma(X \setminus Z, S \otimes I).$$

Proof. (1.a) is an immediate consequence of [Proposition 3.17](#), [Corollary 3.27](#), [Proposition 3.31](#), [Proposition 3.33](#), and [Proposition 3.37](#).

(1.b) is a consequence of [Lemma 3.18](#) and [Proposition 3.25](#).

(2.a) holds by construction and (2.b) follows from [Lemma 3.35 \(2\)](#). \blacksquare

Remark 3.40. If ε is a chirality operator, then S orthogonally decomposes as $\check{S} = \check{S}^+ \oplus \check{S}^-$, $\check{S}^\pm \subset \check{S}$ are Lagrangian subbundles, A preserves this splitting, $\check{H}\Gamma(Z, \check{S})$ orthogonally decomposes as $\check{H}\Gamma(Z, \check{S}) = \check{H}\Gamma(Z, \check{S}^+) \oplus \check{H}\Gamma(Z, \check{S}^-)$, and the residue map restricts to isomorphism

$$\text{res}: \check{H}^\pm \cong \check{H}\Gamma(Z, \check{S}^\pm). \quad \clubsuit$$

3.7 Spectral and local residue conditions

[Theorem 3.39](#) makes it possible to define a wider variety of residue conditions than those considered in [Section 2](#). Here are some examples.

Example 3.41. The APS residue condition is defined by

$$B_{\text{APS}} := \mathbf{1}_{(-\infty, 0)}(A)H^{1/2}\Gamma(Z, \check{S}) \subset \check{H}\Gamma(Z, \check{S});$$

cf. Atiyah, Patodi, and Singer [[APS75](#), (2.3)]. ♠

Proposition 3.42 (Criterion for left semi-Fredholmness). *Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition. If $B \hookrightarrow H^{-1/2}\Gamma(Z, \check{S})$ is compact, then $\text{dom}(D_B) \hookrightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is compact and D_B is left semi-Fredholm.*

Proof. By [Lemma 2.4 \(2\)](#), and the assumption, the composition

$$\begin{aligned} \text{dom}(D_B) &\rightarrow H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \oplus B \hookrightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}) \oplus H^{-1/2}\Gamma(Z, \check{S}) \\ \phi &\mapsto (\phi - \text{ext res}[\phi], \text{res}[\phi]) \end{aligned}$$

is compact. Therefore, by [Theorem 3.39 \(2.b\)](#), $\text{dom}(D_B) \hookrightarrow L^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is compact. Since for every $\phi \in \text{dom}(D_B)$

$$\|\phi\|_D \lesssim \|D\phi\|_{L^2} + \|\phi\|_{L^2},$$

D_B is left semi-Fredholm. ■

Example 3.43. Since $B_{\text{APS}}^G = B_{\text{APS}} \oplus \ker A$, by [Proposition 2.28](#) and [Proposition 3.42](#), $D_{B_{\text{APS}}}$ is Fredholm of index $-\frac{1}{2} \dim \ker A$. In particular, $\dim \ker A$ is even and inherits a symplectic structure from G . If $L \subset \ker A$ is Lagrangian, then $B_{\text{APS}} \oplus L \hookrightarrow H^{-1/2}\Gamma(Z, \check{S})$ is compact and Lagrangian. In particular, [Proposition 2.22](#) applies. ♠

Definition 3.44. Let $V \subset \check{S}$ be a subbundle. The **local residue condition** associated with V is

$$B_V := \check{H}\Gamma(Z, V) \subset \check{H}\Gamma(Z, \check{S}). \quad \bullet$$

Proposition 3.45. *Let $V \subset \check{S}$ be a subbundle. If $V^{\check{\Omega}}$ denotes the symplectic complement of $V \subset \check{S}$, then*

$$B_V^G = B_{V^{\check{\Omega}}}.$$

Proof. This is an immediate consequence of [Proposition 2.17](#) and [Theorem 3.39](#). ■

Example 3.46. Consider the Dirac bundle (S, γ, ∇) corresponding to the Hodge–de Rham operator $d + d^*$; that is: $S := \Lambda T^*X$ with $\gamma(\xi)\phi := \xi \wedge \phi - i_{\xi\#}\phi$. Decompose

$$\check{S} = \check{S}_N \oplus \check{S}_D$$

with $\check{S}_N := S_N \otimes_{\mathbb{C}} NZ^{-1/2}$ and $\check{S}_D := S_D \otimes_{\mathbb{C}} NZ^{-1/2}$, and

$$S_N := (\mathbb{R} \oplus \Lambda^2 N^*Z) \otimes \Lambda T^*Z \quad \text{and} \quad S_D := N^*Z \otimes \Lambda T^*Z.$$

The corresponding residue conditions are Lagrangian. ♠

Example 3.47. Assume that (S, γ, ∇) is a complex Dirac bundle. The MIT bag residue conditions are the local residue conditions arising from the decomposition

$$\check{S} = \check{S}^+ \oplus \check{S}^- \quad \text{with} \quad \check{S}^\pm := \ker(1 \mp iJ);$$

cf. [Joh75]. ♠

Proposition 3.48 (Variation on the bordism theorem). Assume the situation of [Example 3.47](#). The components A^\pm of A in the decomposition

$$A =: \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}$$

satisfy

$$\text{index } A^\pm = 0.$$

Proof. The following proof is essentially identical to the one presented in [BB12, §8.5]. Since $(A^+)^* = A^-$,

$$-\text{index } A^- = \text{index } A^+ = \dim \ker A^+ - \dim \ker A^-.$$

For every $t \in [0, 1]$, set

$$B_t^\pm := \ker A^\pm \oplus (1 \mp tiJ)B_{\text{APS}}.$$

Since $(B_t^+)^G = B_t^-$, by [Proposition 3.42](#), $D_{B_t^\pm}$ is Fredholm. Moreover, by [Proposition 2.28](#),

$$-\text{index } D_{B_t^-} = \text{index } D_{B_t^+} = \text{index } D_{(1 \mp tiJ)B_{\text{APS}}} + \dim \ker A^\pm.$$

In particular,

$$\text{index } A^+ = 2 \text{index } D_{B_0^+}.$$

Therefore, it remains to prove that $\text{index } D_{B_0^+} = 0$. By [Proposition 2.31](#), it suffices to prove that $\text{index } D_{B_1^+} = 0$. Since

$$B_1^\pm = H^{1/2}\Gamma(Z, \check{S}^\pm),$$

analogous to [Example 2.27](#), for every $\phi \in \ker D_{B_1^\pm}$

$$0 = 2\langle D\phi, i\phi \rangle_{L^2} = -2\pi \langle J \text{res } \phi, i \text{res } \phi \rangle = \mp 2\pi \|\text{res } \phi\|_{L^2}^2.$$

Therefore, $\ker D_{B_1^\pm} = \ker D_{\min}$; hence: $\text{index } D_{B_1^\pm} = 0$. ■

4 Regularity theory

This section continues to assume [Hypothesis 3.1](#) throughout. The geometric realisation $\check{H}(Z, \check{S})$ of \check{H} developed in [Section 3](#) and suitable commutator estimates lead to the L^2 regularity theory laid out in the following.

4.1 Adapted Sobolev spaces, I: definition

Here is the scale of Sobolev spaces for which the regularity theory is developed.

Definition 4.1 (Differential operators). Denote by $\text{DiffOp}^\bullet(S \otimes \mathbb{I})$ the \mathbb{N}_0 -filtered ring of differential operators acting on $S \otimes \mathbb{I}$.

- (1) A vector field $v \in \text{Vect}(\hat{X})$ is **conormal** if $v|_{\partial\hat{X}} \in \text{Vect}(\partial\hat{X})$. Denote the subspace of conormal vector fields by $\text{Vect}_b(X \setminus Z)$.
- (2) The filtered subring $\text{DiffOp}_b^\bullet(S \otimes \mathbb{I}) \subset \text{DiffOp}^\bullet(S \otimes \mathbb{I})$ of **conormal** differential operators is generated by $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$ and differential operators of the form ∇_v with $v \in \text{Vect}_b(\hat{X})$.
- (3) The filtered subring $\text{DiffOp}_a^\bullet(S \otimes \mathbb{I}) \subset \text{DiffOp}^\bullet(S \otimes \mathbb{I})$ of **adapted** differential operators is generated by $\text{DiffOp}_b^\bullet(S \otimes \mathbb{I})$ and D . •

Definition 4.2 (Sobolev spaces). Let $k \in \mathbb{N}_0$.

- (b) The **conormal Sobolev space** $H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$ is defined by

$$H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) := \left\{ \phi \in H_{\text{loc}}^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) : \begin{array}{l} P\phi \in L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I}) \text{ for} \\ \text{every } P \in \text{DiffOp}_b^k(S \otimes \mathbb{I}) \end{array} \right\}.$$

Choose a finite subset $\mathcal{P}_b^k \subset \text{DiffOp}_b^k(S \otimes \mathbb{I})$ which spans $\text{DiffOp}_b^k(S \otimes \mathbb{I})$ over $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$. Define the norm $\|\cdot\|_{H_b^k} : H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{H_b^k}^2 := \sum_{P \in \mathcal{P}_b^k} \|P\phi\|_{L^2}^2.$$

- (a) The **adapted Sobolev space** $H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$ is defined by

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) := \left\{ \phi \in H_{\text{loc}}^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) : \begin{array}{l} P\phi \in L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I}) \text{ for} \\ \text{every } P \in \text{DiffOp}_a^k(S \otimes \mathbb{I}) \end{array} \right\}.$$

Choose a finite subset $\mathcal{P}_a^k \subset \text{DiffOp}_a^k(S \otimes \mathbb{I})$ which spans $\text{DiffOp}_a^k(S \otimes \mathbb{I})$ over $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$. Define the norm $\|\cdot\|_{H_a^k} : H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow [0, \infty)$ by

$$\|\phi\|_{H_a^k}^2 := \sum_{P \in \mathcal{P}_a^k} \|P\phi\|_{L^2}^2. \quad \bullet$$

$(H_b^k \Gamma(X \setminus Z, S \otimes \mathbb{I}), \|\cdot\|_{H_b^k})$ and $(H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}), \|\cdot\|_{H_a^k})$ are Hilbert spaces. Evidently, different choices of $\mathcal{P}_b^k, \mathcal{P}_a^k$ lead to equivalent norms. The following discussion leads to particularly convenient choices of $\mathcal{P}_b^k, \mathcal{P}_a^k$.

Definition 4.3 (Convenient vector fields).

- (1) Denote by $\text{Vect}_c(X \setminus Z) \subset \text{Vect}_b(\hat{X})$ the subspace of vector fields supported in $X \setminus Z \subset \hat{X}$.

(2) Denote by $\text{Vect}_{b;c}(\hat{U}) \subset \text{Vect}_b(\hat{X})$ the subspace of vector fields supported in $\hat{U} \subset \hat{X}$. For $v \in \text{Vect}_{b;c}(\hat{U})$, $\mathring{\nabla}_v \in \text{DiffOp}_b^1(S \otimes \mathbb{I})$.

(3) Denote by $\text{Vect}_{b;c,0}(\hat{U})$ the subspace of those $v \in \text{Vect}_{b;c}(\hat{U})$ which are $U(1)$ -invariant on ∂X ; that is: $[\partial_\alpha, v|_{\partial \hat{X}}] = 0$. •

Remark 4.4. $\text{Vect}_{b;c}(\hat{U})$ is generated by $\chi(r) \cdot \partial_\alpha$, $\chi(r) \cdot r \partial_r$, vector field of the form $\chi(r) \cdot v$ where v is lifted from Z , and vector field vanishing near Z . ♣

Lemma 4.5 (Commutation relations). *The following commutation relations hold:*

(1) For every $v \in \text{Vect}_c(X \setminus Z)$, $k \in \mathbb{N}$, and $P \in \text{DiffOp}_a^k(S \otimes \mathbb{I})$

$$[\nabla_v, P] \in \text{DiffOp}_b^k(S \otimes \mathbb{I}).$$

(2) For every $v \in \text{Vect}_{b;c}(\hat{U})$, $k \in \mathbb{N}$, and $P \in \text{DiffOp}_b^k(S \otimes \mathbb{I})$

$$[\mathring{\nabla}_v, P] \in \text{DiffOp}_b^k(S \otimes \mathbb{I}).$$

(3) For every $v \in \text{Vect}_{b;c,0}(\hat{U})$

$$[\mathring{\nabla}_v, D] \in \text{DiffOp}_b^1(S \otimes \mathbb{I}) + \text{DiffOp}_b^0(S \otimes \mathbb{I}) \cdot D.$$

Proof. If $v, w \in \text{Vect}(X \setminus Z)$ and $T \in \Gamma(X \setminus Z, S \otimes \mathbb{I})$, then

$$[\nabla_v, T] = \nabla_v T \quad \text{and} \quad [\nabla_v, \nabla_w] = \nabla_{[v, w]} + F_\nabla(v, w).$$

Therefore, for every $k \in \mathbb{N}$ and $P \in \text{DiffOp}_a^k(S \otimes \mathbb{I})$, $[\nabla_v, P] \in \text{DiffOp}_b^k(S \otimes \mathbb{I})$.

If $v \in \text{Vect}_c(X \setminus Z)$, $k \in \mathbb{N}$, and $P \in \text{DiffOp}_a^k(S \otimes \mathbb{I})$, then $\text{supp}([\nabla_v, P]) \subset X \setminus Z$; therefore and by the above observation, $[\nabla_v, P] \in \text{DiffOp}_b^k(S \otimes \mathbb{I})$. This proves (1).

(2) is immediate from the above observation.

Let $v \in \text{Vect}_{b;c,0}(\hat{U})$. By [Proposition 3.14](#), $D - \chi(r) \cdot \mathring{D} \in \text{DiffOp}_b^1(S \otimes \mathbb{I})$. Therefore, it suffices to prove that

$$[\mathring{\nabla}_v, \chi(r) \cdot \mathring{D}] \in \text{DiffOp}_b^1(S \otimes \mathbb{I}) + \text{DiffOp}_b^0(S \otimes \mathbb{I}) \cdot \chi(r) \cdot \mathring{D}.$$

By direct computation,

$$[\mathring{\nabla}_{r\partial_r}, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = -(\mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}) \quad \text{and} \quad [\mathring{\nabla}_{\partial_\alpha}, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = 0;$$

moreover, if v is the lift of a vector field along Z , then

$$[\mathring{\nabla}_v, \mathring{\nabla}_{\partial_r} - r^{-1}I\mathring{\nabla}_{\partial_\alpha}] = 0.$$

By [Remark 3.13](#), [Remark 4.4](#) and since $\chi(r) \cdot D_Z \in \text{DiffOp}_b^1(S \otimes \mathbb{I})$ and $\chi(r) \cdot J \in \text{DiffOp}_b^0(S \otimes \mathbb{I})$, this implies (3). ■

Corollary 4.6 (Convenient choices of $\mathcal{P}_b^k, \mathcal{P}_a^k$).

- (1) Set $\mathcal{P}_b^0 := \{\text{id}_{S \otimes \mathbb{I}}\}$. For every $k \in \mathbb{N}$ there is a finite subset $\mathcal{P}_b^k \subset \text{DiffOp}_b^k(S \otimes \mathbb{I})$ which spans $\text{DiffOp}_b^k(S \otimes \mathbb{I})$ over $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$ such that $\mathcal{P}_b^{k-1} \subset \mathcal{P}_b^k$ and every $P \in \mathcal{P}_b^k \setminus \mathcal{P}_b^{k-1}$ is of the form

$$P = \nabla_{u_1} \cdots \nabla_{u_\ell} \overset{\circ}{\nabla}_{v_1} \cdots \overset{\circ}{\nabla}_{v_{k-\ell}}$$

with

$$u_1, \dots, u_\ell \in \text{Vect}_c(X \setminus Z) \quad \text{and} \quad v_1, \dots, v_{k-\ell} \in \text{Vect}_{b;c,0}(\hat{U}).$$

- (2) Let $k \in \mathbb{N}$. If $\mathcal{P}_b^1, \dots, \mathcal{P}_b^k$ are as above then

$$\mathcal{P}_a^k := \bigcup_{\ell=0}^k \{PD^\ell : P \in \mathcal{P}_b^{k-\ell}\}$$

spans $\text{DiffOp}_a^k(S \otimes \mathbb{I})$ over $\Gamma(\hat{X}, \text{End}(\hat{S} \otimes \hat{\mathbb{I}}))$. ■

Henceforth, for every $k \in \mathbb{N}$, $\mathcal{P}_a^k, \mathcal{P}_b^k$ are assumed to be chosen as in [Corollary 4.6](#); in particular,

$$(4.7) \quad \|\phi\|_{H_a^k}^2 = \sum_{\ell=0}^k \|D^\ell \phi\|_{H_b^{k-\ell}}^2.$$

Remark 4.8. Let (W, g) be a Riemannian manifold with boundary equipped with a Dirac bundle (S, γ, ∇) . The conormal Sobolev space $H_b^k \Gamma(W, S)$; see, e.g., [\[Mel93, \(5.42\)\]](#); is an important tool in the study of boundary values problems for Dirac operators. A moment's thought with the above discussion in mind shows that $H_a^k \Gamma(W, S)$, the analogue of the adapted Sobolev space, agrees with the usual Sobolev space $H^k \Gamma(W, S)$. Indeed, these are the appropriate Sobolev spaces for the L^2 regularity theory; see [\[BB12, §6.2\]](#). ♣

4.2 Elliptic regularity and estimates

Here is the fundamental regularity result.

Theorem 4.9 (Elliptic regularity and estimates, I). *For every $k \in \mathbb{N}_0$*

$$H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) = \left\{ \phi \in H_{\text{loc}}^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) : \begin{array}{l} \phi, D\phi \in H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \\ \text{and } \text{res}[\phi] \in H^{k+\frac{1}{2}} \Gamma(Z, \check{S}) \end{array} \right\}$$

moreover: for every $\phi \in H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I})$

$$\|\phi\|_{H_a^{k+1}} \asymp_k \|D\phi\|_{H_a^k} + \|\phi\|_{H_a^k} + \|\text{res}[\phi]\|_{H^{k+1/2}}.$$

The proof relies on the following observations.

Lemma 4.10. *For every $k \in \mathbb{N}_0$, the extension map $\text{ext}: \check{H} \Gamma(Z, \check{S}) \rightarrow \text{dom}(D_{\text{max}})$ restricts to a bounded linear map*

$$\text{ext}: H^{k+1/2} \Gamma(Z, \check{S}) \rightarrow H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I});$$

moreover, for every $\phi \in \check{H} \Gamma(Z, \check{S})$, $\text{ext}(\phi) \in H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I})$ if and only if $\phi \in H^{k+1/2} \Gamma(Z, \check{S})$; in that case

$$\|\text{ext} \phi\|_{H_a^{k+1}} \asymp_k \|\phi\|_{H^{k+1/2}}.$$

Proof. By (4.7) and Lemma 4.5, it suffices to prove that for every $(-1/2, \mu) \in \check{\sigma}$ and $\phi \in V_{-1/2, \mu} \subset H^{1/2}\Gamma(Z, \check{S})$

$$\sum_{\ell=0}^{k+1} \|\mathring{D}^\ell \text{ext}(\phi)\|_{H_b^{k-\ell+1}}^2 = \sum_{\ell=0}^{k+1} \|\mathring{D}^\ell(\chi(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{H_b^{k-\ell+1}}^2 \asymp_k \langle \mu \rangle^{2k+1} \|\phi\|^2 \asymp_k \|\phi\|_{H^{k+1/2}}^2.$$

Let f be any bounded smooth function on $(0, 1)$. By direct computation,

$$\begin{aligned} \mathring{\nabla}_{r\partial_r}(f(r)r^{-1/2}e^{-|\mu|r}\phi) &= (r \cdot f'(r) - (\tfrac{1}{2} + |\mu|r) \cdot f(r))r^{-1/2}e^{-|\mu|r}\phi \quad \text{and} \\ \mathring{\nabla}_{\partial_\alpha}(f(r)r^{-1/2}e^{-|\mu|r}\phi) &= \tfrac{1}{2}f(r)r^{-1/2}e^{-|\mu|r}I\phi. \end{aligned}$$

Consequently,

$$\mathring{D}(f(r)r^{-1/2}e^{-|\mu|r}\phi) = (f'(r) - (|\mu| - A_\mu) \cdot f(r))r^{-1/2}e^{-|\mu|r}J\phi.$$

Set

$$\begin{aligned} f_0(r) &:= \chi(r), \quad f_\ell(r) := f'_{\ell-1}(r) - (|\mu| - A_\mu) \cdot f_{\ell-1}(r), \quad \ell \geq 1 \quad \text{and} \\ f_{0,\ell}(r) &:= f_\ell(r), \quad f_{m,\ell}(r) := r \cdot f'_{m-1,\ell}(r) - (\tfrac{1}{2} + |\mu|r) \cdot f_{m-1,\ell}(r), \quad m, \ell \geq 1. \end{aligned}$$

The above implies that

$$(\mathring{\nabla}_{r\partial_r})^m \mathring{D}^\ell(\chi(r)r^{-1/2}e^{-|\mu|r}\phi) = f_{m,\ell}(r)r^{-1/2}e^{-|\mu|r}J^\ell\phi.$$

Therefore,

$$\|(\mathring{\nabla}_{\partial_\alpha})^j (\mathring{\nabla}_{r\partial_r})^m \mathring{D}^\ell(\chi(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{L^2}^2 \lesssim \int_0^1 f_{m,\ell}^2(r)e^{-2|\mu|r}dr \cdot \|\phi\|^2 \lesssim_{\ell,m} \langle \mu \rangle^{2(m+\ell)-1} \cdot \|\phi\|^2.$$

Moreover, if v_1, \dots, v_m are lifts of vector fields along Z , then

$$\begin{aligned} \|\mathring{\nabla}_{v_1} \dots \mathring{\nabla}_{v_m}(f(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{L^2}^2 &\lesssim_{v_1, \dots, v_m} \|(f(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{L^2}^2 + \|(f(r)r^{-1/2}e^{-|\mu|r})A^m\phi\|_{L^2}^2 \\ &\asymp_m \langle \mu \rangle^{2m} \|(f(r)r^{-1/2}e^{-|\mu|r}\phi)\|_{L^2}^2 \\ &\asymp \langle \mu \rangle^{2m} \int_0^1 f^2(r)e^{-2|\mu|r}dr \cdot \|\phi\|^2 \\ &\lesssim \langle \mu \rangle^{2m-1} \|f\|_{L^\infty}^2 \cdot \|\phi\|^2. \end{aligned}$$

This proves the assertion, by Corollary 4.6. ■

Lemma 4.11. *Let $k \in \mathbb{N}_0$ and $P \in \mathcal{P}_b^k$. For every $\phi \in H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I})$ with $\text{res}[\phi] \in H^{k+1/2}\Gamma(Z, \check{S})$, it follows that $\text{res}[P\phi] \in H^{1/2}\Gamma(Z, \check{S})$ with*

$$\|\text{res}[P\phi]\|_{H^{1/2}} \lesssim_P \|\text{res}[\phi]\|_{H^{k+\frac{1}{2}}}.$$

Proof. Evidently,

$$\text{res}[\mathring{\nabla}_{\partial_\alpha}\phi] = \frac{1}{2}I \text{res}[\phi]$$

and for every vector field v along Z ,

$$\text{res}[\mathring{\nabla}_v\phi] = \nabla_v \text{res}[\phi].$$

By [Lemma 4.10](#), $\psi := \phi - \text{ext res}[\phi] \in H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \cap H_a^2\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and therefore

$$(\partial_r - \frac{1}{2r})(r^{1/2} \cdot \mathring{\nabla}_{r\partial_r}\psi) = r^{1/2}(\partial_r + \frac{1}{2r})\mathring{\nabla}_{r\partial_r}\psi - \frac{1}{2}r^{1/2}\mathring{\nabla}_{\partial_r}\psi \in L^2\Gamma(X \setminus Z, S \otimes \mathbb{I}).$$

Consequently, by [Lemma 3.28\(3\)](#), $\text{res}[\mathring{\nabla}_{r\partial_r}\psi] = 0$. Therefore

$$\text{res}[\mathring{\nabla}_{r\partial_r}\phi] = \text{res}[\mathring{\nabla}_{r\partial_r}(\text{ext res}[\phi])] = -\frac{1}{2}\text{res}[\phi].$$

The proof follows. ■

Proof of Theorem 4.9. Let $k \in \mathbb{N}_0$, $\phi \in H_{\text{loc}}^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I}) \cap \text{dom}(D_{\max})$ with $D\phi \in H_a^k\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and $\text{res}[\phi] \in H^{k+1/2}\Gamma(Z, \check{S})$.

Since $H^1\Gamma(X \setminus Z, S \otimes \mathbb{I}) \hookrightarrow H_a^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and by [Lemma 4.10](#),

$$\phi = (\phi - \text{ext res}[\phi]) + \text{ext res}[\phi] \in H_a^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$$

if and only of $\text{res}[\phi] \in H^{1/2}\Gamma(Z, \check{S})$. Moreover,

$$\begin{aligned} \|\phi\|_{H_a^1} &\asymp \|\phi - \text{ext res}[\phi]\|_D + \|\text{res}[\phi]\|_{H^{1/2}} \\ &\asymp \|D\phi\|_{L^2} + \|\phi\|_{L^2} + \|\text{res}[\phi]\|_{H^{1/2}}. \end{aligned}$$

This proves the assertion if $k = 0$.

Suppose that $k \in \mathbb{N}$. If $\phi \in H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I})$, then, by the above,

$$\begin{aligned} \|\text{res}[\phi]\|_{H^{k+1/2}} &\lesssim_k \|\text{res}[\phi]\|_{H^{1/2}} + \|A^k \text{res}[\phi]\|_{H^{1/2}} \\ &\lesssim \|\text{res}[\phi]\|_{H^{1/2}} + \|\text{res}[\chi(r) \cdot A^k \phi]\|_{H^{1/2}} \\ &\lesssim \|\phi\|_{H_a^1} + \|\chi(r) \cdot A^k \phi\|_{H_a^1} \lesssim_k \|\phi\|_{H_a^{k+1}}. \end{aligned}$$

If $P \in \mathcal{P}_b^k$, then, by [Lemma 4.5](#),

$$\|DP\phi\|_{L^2} \lesssim_P \|D\phi\|_{H_a^k} + \|\phi\|_{H_a^k};$$

moreover: $\|\text{res } P\phi\|_{H^{1/2}} \lesssim_P \|\text{res}[\phi]\|_{H^{k+1/2}}$, by [Lemma 4.11](#). Therefore, if $\text{res}[\phi] \in H^{k+1/2}\Gamma(Z, S \otimes \mathbb{I})$, then by the above, $P\phi \in H_a^1\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and

$$\|P\phi\|_{H_a^1} \lesssim_P \|D\phi\|_{H_a^k} + \|\phi\|_{H_a^k} + \|\text{res}[\phi]\|_{H^{k+1/2}}.$$

This implies the assertion. ■

For suitable residue conditions $B \subset \check{H}\Gamma(Z, \check{S})$, the term $\text{res}[\phi]$ in [Theorem 4.9](#) can be absorbed provided $\text{res}[\phi] \in B$.

Definition 4.12. Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition.

- (1) Let $k \in \mathbb{N}_0$. B is $(k + \frac{1}{2})$ -regular if for every $\phi \in B$

$$\|\phi\|_{H^{k+1/2}} \lesssim_{B,k} \|\mathbf{1}_{(-\infty, 0)}(A)\phi\|_{H^{k+1/2}} + \|\phi\|_{\check{H}}.$$

- (2) B is ∞ -regular if it is $(k + 1/2)$ -regular for every $k \in \mathbb{N}_0$. •

Example 4.13. The APS residue condition B_{APS} is ∞ -regular. ♠

Theorem 4.14 (Elliptic regularity and estimates, II). *Let $k \in \mathbb{N}_0$. Let B be a $(k + \frac{1}{2})$ -regular residue condition. If $\phi \in H_{\text{loc}}^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I})$ satisfies $\phi, D\phi \in H_a^k\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and $\text{res}[\phi] \in B$, then $\phi \in H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I})$ and*

$$\|\phi\|_{H_a^{k+1}} \asymp_{B,k} \|D\phi\|_{H_a^k} + \|\phi\|_{L^2}.$$

The proof requires the following preparation.

Lemma 4.15. *For every $k \in \mathbb{N}_0$ and $\phi \in \text{dom}(D_{\text{max}})$*

$$\|\mathbf{1}_{(-\infty, 0)}(A) \text{res}[\phi]\|_{H^{k+1/2}} \lesssim_k \|D\phi\|_{H_b^k} + \|\phi\|_{H_b^k}.$$

Proof. Since $\text{res} \circ [\cdot] : \text{dom}(D_{\text{max}}) \rightarrow \check{H}\Gamma(Z, \check{S})$ is bounded and $\chi(r) \cdot A^k \in \text{DiffOp}_b^k(S \otimes \mathbb{I})$,

$$\begin{aligned} \|\mathbf{1}_{(-\infty, 0)}(A) \text{res}[\phi]\|_{H^{k+1/2}} &\lesssim_k \|A^k \text{res}[\phi]\|_{\check{H}} = \|\text{res}[\chi(r) \cdot A^k \phi]\|_{\check{H}} \\ &\lesssim \|\chi(r) \cdot A^k \phi\|_D \lesssim_k \|D\phi\|_{H_b^k} + \|\phi\|_{H_b^k} \end{aligned}$$

by [Corollary 4.6](#). ■

Proof of Theorem 4.14. By [Lemma 4.15](#) and since B is $(k + \frac{1}{2})$ -regular, for every $\phi \in \text{dom}(D_B)$

$$\|\text{res}[\phi]\|_{H^{k+1/2}} \lesssim_{B,k} \|\mathbf{1}_{(-\infty, 0)}(A) \text{res}[\phi]\|_{H^{k+1/2}} + \|\text{res}[\phi]\|_{\check{H}} \lesssim_k \|D\phi\|_{H_b^k} + \|\phi\|_{H_b^k}.$$

This together with [Theorem 4.14](#) implies the assertion. ■

4.3 Fredholm extensions in higher regularity

The following is a consequence of [Proposition 3.42](#).

Corollary 4.16 ($\frac{1}{2}$ -regular implies left semi-Fredholm). *Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition. If B is $\frac{1}{2}$ -regular, then D_B is left semi-Fredholm.* ■

The discussion in [Section 4.2](#) leads to the following observation.

Definition 4.17. Let $k \in \mathbb{N}_0$. Let $B \subset \check{H}\Gamma(Z, \check{S})$ be a residue condition. Consider the closed subspace

$$H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I}; B) := \{\phi \in H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I}) : \text{res}[\phi] \in B\}$$

and the restriction of D to

$$D_{B,k} : H_a^{k+1}\Gamma(X \setminus Z, S \otimes \mathbb{I}; B) \rightarrow H_a^k\Gamma(X \setminus Z, S \otimes \mathbb{I}). \quad \bullet$$

Proposition 4.18. *Let $k \in \mathbb{N}_0$. Let $B \subset \check{H}\Gamma(Z, \check{S})$ is a $(k + \frac{1}{2})$ -regular residue condition, then $D_{B,k}$ is left semi-Fredholm; in fact:*

$$\ker D_{B,k} = \ker D_B$$

and the canonical map

$$\operatorname{coker} D_{B,k} \rightarrow \operatorname{coker} D_B \cong (\ker D_{B^G})^*$$

is an isomorphism; moreover, if $\ker D_{B^G} \subset H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$, then the latter L^2 orthogonally decomposes as

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) = \operatorname{im} D_{B,k} \oplus \ker D_{B^G}.$$

Proof. The proof is identical to the one of [DW24, Theorem 3.57], but repeated here for the readers' convenience. By Theorem 4.14, $\ker D_{B,k} = \ker D_B$; moreover: the linear map

$$\frac{\operatorname{dom}(D_B)}{H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}; B)} \rightarrow \frac{L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I})}{H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})}$$

induced by D_B is injective. Therefore, by the Snake Lemma, the canonical map

$$\operatorname{coker} D_{B,k} \rightarrow \operatorname{coker} D_B$$

is injective.

Since $H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$ is dense in $L^2 \Gamma(X \setminus Z, S \otimes \mathbb{I})$ the map

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow (\ker D_{B^G})^* \cong \operatorname{coker} D_B$$

is surjective. Since it factors through $\operatorname{coker} D_{B,k} \rightarrow \operatorname{coker} D_B$, the latter must be surjective. ■

Proposition 4.19. *For every $k \in \mathbb{N}_0$ the restriction of D to*

$$D_k : H_a^{k+1} \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$$

is right semi-Fredholm; that is: $\operatorname{im} D_k$ is closed and $\operatorname{coker} D_k$ is finite-dimensional; moreover: $H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$ L^2 orthogonally decomposes as

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) = \operatorname{im} D_k \oplus \ker D_{\min}.$$

Proof. Since $\ker D_{B_{\text{APS}}}$ is finite-dimensional, there is a $\tau \leq 0$ such that the projection

$$\operatorname{res}(\ker D_{B_{\text{APS}}}) \rightarrow \bigoplus_{\lambda \in [\tau, 0)} \ker(A - \lambda \mathbf{1})$$

is injective. The residue conditions

$$B_\tau := \mathbf{1}_{(-\infty, \tau)}(A) H^{1/2} \Gamma(Z, \check{S}) \subset \check{H}\Gamma(Z, \check{S})$$

and B_τ^G are ∞ -regular. Therefore, by Proposition 4.18,

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) = \operatorname{im} D_{B_\tau^G, k} \oplus \ker D_{B_\tau, k}.$$

By construction, $\ker D_{B_\tau, k} = \ker D_{\min}$. Moreover, since $\operatorname{im} D_k \perp \ker D_{\min}$, $\operatorname{im} D_{B_\tau^G, k} = \operatorname{im} D_k$. ■

4.4 Adapted Sobolev spaces, II: Morrey embedding and polyhomogeneity

The purpose of the upcoming two subsections is to further understand the scale of adapted Sobolev spaces $(H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}), \|\cdot\|_{H_a^k})_{k \in \mathbb{N}_0}$. A crucial observation is that the singularities in the elements of $H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$ can be removed after untwisting in the following sense.

Definition 4.20. Define the **twist** $\bar{z}^{-1/2} \in \Gamma(U \setminus Z, \text{Hom}_{\mathbb{C}}(\Pi^* \check{S}, \mathring{S} \otimes \mathring{\mathbb{I}}))$ by

$$\bar{z}^{-1/2} \phi := r^{-1/2} P_{-1/2} \phi$$

with $P_{-1/2}$ as in [Proposition 3.21](#). •

Lemma 4.21 (Removable singularities after untwisting). *For every $k \in \mathbb{N}_0$ the restriction map $H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow H_{\text{loc}}^k \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})$ factors through*

$$\bar{z}^{-1/2} H^k \Gamma(U, \Pi^* \check{S}) \subset H_{\text{loc}}^k \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}).$$

Proof. Let $k \in \mathbb{N}_0$ and $\phi \in H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I})$. Set

$$\psi := \bar{z}^{1/2} \phi \in H_{\text{loc}}^k \Gamma(U \setminus Z, \Pi^* \check{S}).$$

Since $|\bar{z}^{1/2}|$ is bounded, $\psi \in r^{1/2} L^2 \Gamma(U, \Pi^* \check{S})$. This proves the assertion for $k = 0$.

Henceforth, suppose that $k \in \mathbb{N}$. Consider the differential operators $\mathfrak{d}_z : H_{\text{loc}}^1 \Gamma(U, \Pi^* \check{S}) \rightarrow L^2 \Gamma(U, \Pi^* \check{S})$ and $\mathring{\mathfrak{d}}_z : H_{\text{loc}}^1 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}}) \rightarrow L^2 \Gamma(U \setminus Z, \mathring{S} \otimes \mathring{\mathbb{I}})$ defined by

$$\mathfrak{d}_z := J(\partial_r - r^{-1} I \nabla_{\partial_\alpha}) \quad \text{and} \quad \mathring{\mathfrak{d}}_z := J(\partial_r - r^{-1} I \mathring{\nabla}_{\partial_\alpha}).$$

The difference $\mathring{D} - \mathring{\mathfrak{d}}_z$ is a first order conormal differential operator; see the proof of [Lemma 4.5](#). If v is the lift of a vector field on Z , then

$$\nabla_v \psi = \bar{z}^{1/2} \mathring{\nabla}_v \phi \quad \text{and} \quad \mathfrak{d}_z \psi = \bar{z}^{1/2} \mathring{\mathfrak{d}}_z \phi$$

on $U \setminus Z$.

A moment's thought shows that $\nabla_v \psi = \bar{z}^{1/2} \mathring{\nabla}_v \phi$ holds on U in the sense of distributions. In fact, $\mathfrak{d}_z \psi = \bar{z}^{1/2} \mathring{\mathfrak{d}}_z \phi$ also holds on U in the sense of distributions. To see this let η_ε be a suitable cut-off function and τ a test section. By direct computation,

$$\int_U \langle \eta_\varepsilon \tau, \mathfrak{d}_z \psi \rangle = \int_U \langle \eta_\varepsilon \mathfrak{d}_z^* \tau, \psi \rangle + \langle \sigma_{\mathfrak{d}_z}(\text{d}\eta_\varepsilon) \tau, \psi \rangle$$

and

$$\left| \int_U \langle \sigma_{\mathfrak{d}_z}(\text{d}\eta_\varepsilon) \tau, \psi \rangle \right| \lesssim_\tau \int_U r^{1/2} |\text{d}\eta_\varepsilon| r^{-1/2} |\psi| \lesssim_\psi \int_U r |\text{d}\eta_\varepsilon|^2.$$

Since η_ε can be chosen so that $\text{d}\eta_\varepsilon$ is supported in $B_{2\varepsilon}(Z)$ and $r |\text{d}\eta_\varepsilon| \lesssim 1$, it follows that

$$\int_U \langle \tau, \mathfrak{d}_z \psi \rangle - \langle \mathfrak{d}_z^* \tau, \psi \rangle = \lim_{\varepsilon \downarrow 0} \int_U \eta_\varepsilon (\langle \tau, \mathfrak{d}_z \psi \rangle - \langle \mathfrak{d}_z^* \tau, \psi \rangle) = 0.$$

By induction, it follows that if $v_1, \dots, v_{k-\ell}$ are lifts of vector fields on Z , then

$$\nabla_{v_1} \dots \nabla_{v_{k-\ell}} \mathfrak{d}_z^\ell \psi = \bar{z}^{1/2} \overset{\circ}{\nabla}_{v_1} \dots \overset{\circ}{\nabla}_{v_{k-\ell}} \overset{\circ}{\mathfrak{d}}_z^\ell \phi$$

holds on U in the sense of distributions; in particular:

$$\nabla_{v_1} \dots \nabla_{v_{k-\ell}} \mathfrak{d}_z^\ell \psi \in r^{1/2} L^2 \Gamma(U, \Pi^* \check{S}).$$

This implies that $\psi \in H^k \Gamma(U, \Pi^* \check{S})$ because

$$\int_U |\mathfrak{d}_z(\chi\psi)|^2 = \int_U |\partial_r(\chi\psi)|^2 + r^{-2} |\nabla_{\partial_\alpha}(\chi\psi)|^2 + O(|\psi|^2). \quad \blacksquare$$

Set

$$H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I}) := \bigcap_{k \in \mathbb{N}_0} H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}).$$

Corollary 4.22 (Smooth after untwisting). *The restriction map $H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I}) \rightarrow \Gamma(U \setminus Z, \check{S} \otimes \overset{\circ}{\mathbb{I}})$ factors through*

$$\bar{z}^{-1/2} \Gamma(U, \Pi^* \check{S}) \subset \Gamma(U \setminus Z, \check{S} \otimes \overset{\circ}{\mathbb{I}}). \quad \blacksquare$$

Remark 4.23 (Polyhomogeneous expansion). Every $\psi \in \Gamma(U, \Pi^* \check{S})$ has a Taylor expansion

$$\psi \sim \sum_{k, \ell=0}^{\infty} \bar{z}^k z^\ell \check{\psi}_{k, \ell} \quad \text{with} \quad \check{\psi}_{k, \ell} \in \Pi^* \Gamma(Z, \check{S} \otimes_{\mathbb{C}} NZ^{k-\ell})$$

at Z . Here $\bar{z} \in \Gamma(U, \text{Hom}_{\mathbb{C}}(\Pi^* NZ, \mathbb{C}))$ and $z \in \Gamma(U, \text{Hom}_{\mathbb{C}}(\Pi^* NZ^{-1}, \mathbb{C}))$ denote the tautological sections. Therefore, by [Corollary 4.22](#), every $\phi \in H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I})$ has a polyhomogeneous expansion

$$\phi \sim \sum_{k, \ell=0}^{\infty} \bar{z}^{k-1/2} z^\ell \check{\phi}_{k, \ell} \quad \text{with} \quad \check{\phi}_{k, \ell} \in \Pi^* \Gamma(Z, \check{S} \otimes_{\mathbb{C}} NZ^{k-\ell})$$

at Z with $\bar{z}^{k-1/2} := \bar{z}^k \bar{z}^{-1/2}$. Moreover, a moment's thought shows that if $D\phi = 0$, then the leading order term is of the form $\bar{z}^{k-1/2} \check{\phi}_{k, 0}$ for some $k \in \mathbb{N}_0$. \clubsuit

Corollary 4.24. *For every $k \in \mathbb{N}$ with $k \geq n/2$*

$$H_a^k \Gamma(X \setminus Z, S \otimes \mathbb{I}) \subset r^{-1/2} L^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I}). \quad \blacksquare$$

The above observation leads to the following “poor man’s Weyl law”.

Proposition 4.25 (Growth of eigenvalues). *Let $k \in \mathbb{N}_0$ with $k > n/2$. If $B \subset \check{H} \Gamma(X \setminus Z, S \otimes \mathbb{I})$ is a $(k + \frac{1}{2})$ -regular Lagrangian residue condition, then the **counting function** $N: [0, \infty) \rightarrow \mathbb{N}_0$ defined by*

$$N(\Lambda) := \dim E_{\leq \Lambda} \quad \text{with} \quad E_{\leq \Lambda} := \bigoplus_{\lambda \in [-\Lambda, \Lambda]} \ker(D_B - \lambda \cdot \mathbf{1})$$

satisfies

$$N(\Lambda) \lesssim_{B, k} \langle \Lambda \rangle^{2k}.$$

Proof. The following argument is due to Li [Li80, Lemma 11]. Choose an L^2 orthonormal basis $(\phi_1, \dots, \phi_{N(\Lambda)})$ of $E_{\leq \Lambda}$. The density $d \in C^\infty(X \setminus Z, [0, \infty))$ defined by

$$d := \sum_{i=1}^{N(\Lambda)} |\phi_i|^2$$

does not depend on the choice of L^2 orthonormal basis. By construction

$$N(\Lambda) = \frac{1}{\text{vol}(X)} \int_X d \lesssim \|rd\|_{L^\infty}.$$

Choose an $x \in X$ with $\|rd\|_{L^\infty} \leq 2|rd|(x)$. Since $\text{ev}_x: E_{\leq \Lambda} \rightarrow (S \otimes \mathbb{I})_x$ has rank at most $\text{rk } S$, without loss of generality,

$$|rd|(x) = \sum_{i=1}^{\text{rk } S} r|\phi_i|^2(x).$$

By [Theorem 4.14](#) and [Corollary 4.24](#)

$$\|r^{1/2}\phi_i\|_{L^\infty} \lesssim \|\phi_i\|_{H_a^k} \lesssim_{B,k} \|D^k \phi_i\|_{L^2} + \|\phi_i\|_{L^2} \lesssim \langle \Lambda \rangle^k.$$

This implies the assertion. ■

Corollary 4.26. *Let $k \in \mathbb{N}_0$ with $k > n/2$. If $B \subset \check{H}\Gamma(X \setminus Z, S \otimes \mathbb{I})$ is a $(k + \frac{1}{2})$ -regular Lagrangian residue condition, then for every $t \in (0, \infty)$ the heat operator $h_t := \exp(-tD_B^2)$ is trace class.* ■

Remark 4.27. Assume the situation of [Corollary 4.26](#). If ε is a chirality operator, then for every $t > 0$

$$\text{index } D_B^+ = \dim \ker D_B^+ - \dim \ker D_B^- = \text{str}_\varepsilon h_t.$$

Here str_ε denotes the super trace with respect to ε of the heat operator h_t ; cf. [BGV92, §1.3]. For suitable choices of B an analysis of the asymptotic behaviour of the kernel attached to h_t as $t \downarrow 0$ should result in index formulae analogous to the one established by Atiyah, Patodi, and Singer [APS75, Theorems 3.10 and 4.2]. It would be interesting to work this out in detail. Also, it should be mentioned that part of the unpublished PhD thesis [Yano7, Theorems 1.0.3 and 2.3.4] discusses such index formulae. ♣

4.5 Adapted Sobolev spaces, III: spectral description and tameness

Proposition 4.28. *The graded Fréchet space $(H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I}), (\|\cdot\|_{H_a^k})_{k \in \mathbb{N}_0})$ is tame.*

Proof. Consider the ∞ -regular Lagrangian residue condition $B := B_{\text{APS}} \oplus L$, where $L \subset \ker A$ is a Lagrangian subspace, as described in [Example 3.43](#). The operator D_B is self-adjoint and Fredholm and [Proposition 2.22](#) applies. The graded Fréchet space $(H_a^\infty \Gamma(X \setminus Z, S \otimes \mathbb{I}; B), (\|\cdot\|_{H_a^k})_{k \in \mathbb{N}_0})$ is tame in the sense of [Ham82, Part II Definition 1.3.2] can be seen as follows. Consider the graded

Fréchet space $(\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})), (\|-\|_k)_{k \in \mathbb{N}_0})$ of exponentially decreasing sequences defined by

$$\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})) := \{(\psi_\beta) \in L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})^{\mathbb{N}_0} : \|\psi_\beta\|_k < \infty \text{ for every } k \in \mathbb{N}_0\}$$

with

$$\|(\psi_\beta)\|_k^2 := \sum_{\beta=0}^{\infty} e^{2k\beta} \|\psi_\beta\|_{L^2}^2;$$

cf. Hamilton [Ham82, Part II Example 1.1.4(b) with $q = 2$].

Let $(\phi_\alpha)_{\alpha \in \mathbb{N}_0}$ be an L^2 orthonormal basis of $L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})$ consisting of eigenspinors for D_B and denote by $(\lambda_\alpha)_{\alpha \in \mathbb{N}_0}$ the corresponding sequence of eigenvalues. Define $i: H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}; B) \rightarrow \Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathbb{I}))$ and $p: \Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})) \rightarrow H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}; B)$ by

$$(i\phi)_\beta := \sum_{\alpha=0}^{\infty} \mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle \lambda_\alpha \rangle) \langle \phi, \phi_\alpha \rangle_{L^2} \phi_\alpha \quad \text{and} \quad p(\psi)_\beta := \sum_{\alpha, \beta=0}^{\infty} \mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle \lambda_\alpha \rangle) \langle \psi_\beta, \phi_\alpha \rangle_{L^2} \phi_\alpha.$$

A moment's thought shows that $p \circ i = \text{id}$; moreover, by Theorem 4.14:

$$\|i(\phi)\|_k^2 = \sum_{\beta=0}^{\infty} e^{2k\beta} \|i(\phi)_\beta\|_{L^2}^2 \leq \sum_{\alpha=0}^{\infty} \langle \lambda_\alpha \rangle^{2k} \langle \phi, \phi_\alpha \rangle_{L^2}^2 \lesssim_k \|\phi\|_{H_a^k}$$

and

$$\begin{aligned} \|p(\psi)_\beta\|_{H_a^k}^2 &\lesssim_k \sum_{\alpha=0}^{\infty} \langle \lambda_\alpha \rangle^{2k} \langle p(\psi)_\beta, \phi_\alpha \rangle^2 \\ &= \sum_{\alpha, \beta=0}^{\infty} \langle \lambda_\alpha \rangle^{2k} \mathbf{1}_{[e^\beta, e^{\beta+1})}(\langle \lambda_\alpha \rangle) \langle \psi_\beta, \phi_\alpha \rangle^2 \leq e^{2k} \sum_{\beta=0}^{\infty} e^{2k\beta} \|\psi_\beta\|_{L^2}^2. \end{aligned}$$

Therefore, $(H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}; B), (\|-\|_{H_a^k})_{k \in \mathbb{N}_0})$ is a tame direct summand in the sense of [Ham82, Part II Definition 1.3.1] of $(\Sigma(L^2\Gamma(X\backslash Z, S \otimes \mathbb{I})), (\|-\|_k)_{k \in \mathbb{N}_0})$.

A similar argument, using an L^2 orthonormal basis of JB consisting of eigenspinors for A in the definition of i and p , proves that the graded Fréchet space $(JB \cap \Gamma(Z, \check{S}), (\|-\|_{H^k})_{k \in \mathbb{N}_0})$ is a tame direct summand of $(\Sigma(L^2\Gamma(Z, \check{S})), (\|-\|_k)_{k \in \mathbb{N}_0})$.

Finally, the tameness of the above two spaces implies that the graded Fréchet space $(H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}), (\|-\|_{H_a^k})_{k \in \mathbb{N}_0})$ is tame, due to the following tame isomorphism:

$$H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}) \cong H_a^\infty\Gamma(X\backslash Z, S \otimes \mathbb{I}; B) \oplus (JB \cap \Gamma(Z, \check{S})),$$

given by $\phi \mapsto (\phi - \text{ext res}_B[\phi], \text{res}_{JB}[\phi])$, where $\text{res} = \text{res}_B \oplus \text{res}_{JB}$ is the canonical decomposition. Indeed, the tameness of this isomorphism follows from ?? of Lemma 4.10 and Theorem 4.9. ■

4.6 Symbolic criterion for ∞ -regularity

Proposition 4.29 (symbolic criterion for ∞ -regularity). *Let $V \subset \check{S}$ be a subbundle. If*

$$\gamma(\xi)V \subset V^\perp = JV^\perp$$

*for every $\xi \in T^*Z \setminus \{0\}$, then B_V is ∞ -regular.*

Proof. The proof relies on the following observation which is already implicit in [FS98, Lemma 2]. Denote by $\text{pr}_V : \check{S} \rightarrow \check{S}$ the orthogonal projection onto V . The operator $\text{pr}_V \text{Apr}_V$ is a 0-th order differential operator, because for every $f \in C^\infty(Z)$

$$[\text{pr}_V \text{Apr}_V, f] = -\text{pr}_V J\gamma(df)\text{pr}_V = 0.$$

Let $k \in \mathbb{N}_0$. Since $\text{pr}_V \text{Apr}_V$ is a 0-th order differential operator, $\text{pr}_V A^{2k+1} \text{pr}_V$ is a $2k$ -th order differential operator. Therefore, for every $\phi \in \Gamma(Z, V)$ and $\varepsilon > 0$

$$\begin{aligned} \langle A^{2k+1} \phi, \phi \rangle_{L^2} &= \langle \text{pr}_V A^{2k+1} \text{pr}_V \phi, \phi \rangle_{L^2} \lesssim_{B_V, k} \|\phi\|_{H^k}^2 \\ &\lesssim \|\phi\|_{H^{k-1/2}} \|\phi\|_{H^{k+1/2}} \lesssim \varepsilon^{-1} \|\phi\|_{H^{k-1/2}}^2 + \varepsilon \|\phi\|_{H^{k+1/2}}^2; \end{aligned}$$

moreover, by direct inspection,

$$\langle A^{2k+1} \phi, \phi \rangle_{L^2} = -\| |A|^{k+1/2} \mathbf{1}_{(-\infty, 0)}(A) \phi \|_{L^2}^2 + \| |A|^{k+1/2} \mathbf{1}_{[0, \infty)}(A) \phi \|_{L^2}^2.$$

As a consequence, for every $\phi \in B_V$,

$$\|\phi\|_{H^{k+1/2}}^2 \lesssim_{B_V, k} \|\mathbf{1}_{(-\infty, 0)}(A) \phi\|_{H^{k+1/2}}^2 + \|\phi\|_{H^{k-1/2}}^2.$$

By induction, B_V is $(k + 1/2)$ -regular for every $k \in \mathbb{N}_0$; hence: ∞ -regular. ■

Corollary 4.30. *Let $V \subset \check{S}$ be a subbundle. If*

$$\gamma(\xi)V = JV^\perp$$

*for every $\xi \in T^*Z \setminus \{0\}$, then B_V and B_V^G are ∞ -regular and Fredholm.* ■

Example 4.31. The local residue conditions defined in [Example 3.46](#) satisfy the criterion in [Corollary 4.30](#). ♠

Example 4.32. If

$$\mathbf{L} \in \Gamma(Z, \text{Hom}_{\mathbb{C}}(\overline{NZ}, \check{S}))$$

is nowhere-vanishing, then

$$V := \text{im } \mathbf{L} \subset \check{S}$$

is a rank one complex subbundle. Therefore and since $J\gamma(\xi)$ and $IJ\gamma(\xi)$ are skew-adjoint, $\gamma(\xi)V \subseteq JV^\perp$ for every $\xi \in T^*Z \setminus \{0\}$; that is: V satisfies the criterion in [Proposition 4.29](#). Moreover: if $\text{rk}_{\mathbb{C}} \check{S} = 2$, then $J\gamma(\xi)V = V^\perp$ and B_V is self-adjoint. ♠

Remark 4.33. Suppose that $(Z, I; \Phi)$ is a harmonic $\mathbb{Z}/2\mathbb{Z}$ spinor whose branching locus Z satisfies [Hypothesis 3.1](#). Since $\Phi \in \ker D_{\min}$, by [Remark 4.23](#), the polyhomogeneous expansion of Φ at Z is of the form

$$\Phi \sim \bar{z}^{1/2} \check{\Phi}_{1,0} + \sum_{\substack{k,\ell=0 \\ k+\ell \geq 2}}^{\infty} \bar{z}^{k-1/2} z^{\ell} \check{\Phi}_{k,\ell}.$$

The leading coefficient $\check{\Phi}_{1,0}$ determines an $L_{\Phi} \in \Gamma(Z, \text{Hom}_{\mathbb{C}}(\overline{NZ}, \check{S}))$. If $\text{rk } S = 4$ and $\check{\Phi}_{1,0}$ is nowhere vanishing, then this produces an ∞ -regular Lagrangian local residue condition. This is the residue condition behind the scenes in [\[Tak15; Par23\]](#). ♣

Remark 4.34. In the presence of a chirality operator ε the above discussion refines as follows:

- (1) Let $V^+ \subset \check{S}^+$ be subbundle. If

$$\gamma(\xi)V^+ = J(V^+)^{\perp} \cap \check{S}^+$$

for every $\xi \in T^*Z \setminus \{0\}$, then $V := V^+ \oplus V^-$ with $V^- := J(V^+)^{\perp} \cap \check{S}^-$ satisfies the condition in [Corollary 4.30](#).

- (2) If $\text{rk } S^+ = 4$ and

$$L \in \Gamma(Z, \text{Hom}_{\mathbb{C}}(\overline{NZ}, \check{S}^+))$$

is nowhere vanishing, then $V^+ := \text{im } L$ satisfies the above condition.

- (3) As in [Remark 4.33](#), the leading coefficient of the polyhomogeneous expansion of a positive harmonic $\mathbb{Z}/2\mathbb{Z}$ spinor Φ determines a $L_{\Phi} \in \Gamma(Z, \text{Hom}_{\mathbb{C}}(\overline{NZ}, \check{S}^+))$. ♣

A Non-coorientable branching loci

The following discussion explains what changes need to be made in [Section 3](#) and [Section 4](#) if $Z \subset X$ is a closed submanifold of codimension two, but not cooriented or even coorientable.

The **coorientation bundle**

$$\mathfrak{o} := \Lambda^2 NZ \rightarrow Z$$

is a Euclidean line bundle and its unique orthogonal connection is flat. The Euclidean metric on NZ identifies \mathfrak{o} with the bundle of skew-adjoint endomorphisms of NZ . The evaluation map defines an isometry

$$I: \mathfrak{o} \otimes NZ \rightarrow NZ.$$

A moment's thought shows that the diagram

$$\begin{array}{ccccccc} NZ & \xrightarrow{\cong} & \mathfrak{o}^{\otimes 2} \otimes NZ & \xrightarrow{1 \otimes I} & \mathfrak{o} \otimes NZ & \xrightarrow{I} & NZ \\ & & & & & \searrow & \nearrow \\ & & & & & -1 & \end{array}$$

commutes. A trivialisation $\mathfrak{o} \cong \mathbb{R}$ enhances I to an orthogonal almost complex structure on NZ ; that is: a coorientation of Z enhances NZ to a Hermitian line bundle.




The canonical isomorphism $\mathfrak{o}^{\otimes 2} \cong \mathbf{R}$ produces the flat bundle

$$\mathbf{A} := \mathbf{R} \oplus i \cdot \mathfrak{o} \rightarrow Z$$



of normed \mathbf{R} -algebras whose fibres are isomorphic to \mathbf{C} , canonically up to complex conjugation. The above discussion reveals NZ to be a bundle of Euclidean \mathbf{A} -modules of rank one. Systematically replacing \mathbf{C} by \mathbf{A} and tracking the use of \mathfrak{o} in [Section 3](#) and [Section 4](#) removes the need for a coorientation of Z :


- The frame bundle $\pi: F \rightarrow Z$ defined in [Definition 3.4](#) is not $U(1)$ -principal. Its vertical tangent bundle $\ker T\pi$ is canonically isomorphic to $i\pi^*\mathfrak{o}$. Therefore, the Levi-Civita connection defines $i\theta \in \Omega^1(F, i\pi^*\mathfrak{o})$; moreover, $\partial_\alpha \in \Gamma(F, \pi^*(\mathfrak{o} \otimes NZ))$.
- [Definition 3.8](#) reveals $S|_Z$ to be an \mathbf{A} -module and defines $J \in \Gamma(F, \text{End}(\underline{S}))$ and $I, K = IJ \in \Gamma(F, \pi^*\mathfrak{o} \otimes \text{End}(\underline{S}))$. The sign ambiguities in the term $I\check{\nabla}_{\partial_\alpha}$ appearing in [Remark 3.13](#), [Proposition 3.22](#), and the proof of [Lemma 4.21](#) cancel.
- [Definition 3.19](#) constructs NZ^λ as an \mathbf{A} -module. [Remark 3.20](#), [Proposition 3.21](#), [Proposition 3.22](#) hold with \mathbf{C} replaced by \mathbf{A} . This can be seen, e.g., by passing to the double cover $\tilde{Z} \rightarrow Z$ defined by \mathfrak{o} .
- In the definition of the residue bundle \check{S} and the branching locus operator A in [Definition 3.36](#) the appearances of \mathbf{C} need to be replaced by \mathbf{A} . \check{S} inherits $J \in \Gamma(Z, \text{End}(\check{S}))$ and $I, K = IJ \in \Gamma(Z, \mathfrak{o} \otimes \text{End}(\check{S}))$.
- \mathbf{C} needs to be replaced by \mathbf{A} in [Example 3.46](#), [Definition 4.20](#), [Remark 4.23](#).

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